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# Practical Approximations for the Electric Field of Elliptical Beams

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**Abstract**

We develop a general approximation scheme to represent the electric field produced by a given charge distribution, such that the approximate expression is accurate both at short and long distances to any desired order, and interpolates smoothly between these limits. The field is given in the form of a ratio of two complex polynomials. This form is fast to compute, is manifestly singularity-free in the round-beam limit, and is well suited for simulations involving space-charge effects. We illustrate the method in the two-dimensional case by applying it to a coasting beam of elliptical Gaussian profile through octupole order.

## 1 Introduction

In dealing with the effects of the space-charge force in particle beam studies one is usually confronted with the calculation of the electric field of a complicated charge density. Few applications involve charge densities with spherical or cylindrical symmetry, and even fewer yield an analytic solution for the field. Furthermore, a class of problems requires self-consistent calculations, in which the charge density evolves in time in response to the external focusing forces in addition to the very space-charge force which it produces. In addition, computational speed is almost always a crucial necessity. Possible solutions include making simplifying assumptions about the charge density to allow an analytic solution for the field [1, 2], approximate expressions for the field itself [3], Lie-algebraic methods [4], etc.

In this note we take a compromise approach, by which we provide an approximate expression for the electric field of a charge distribution which is only known incompletely. To be precise, imagine a localized distribution centered at the origin. Given the first $N$ $k$-space moments and the first $M$ multipole coefficients of this distribution, we provide an approximate expression for the field such that its Taylor expansion at the origin is exact to order $x^{N-1}$, and its expansion at infinity is exact to order $x^{-M}$. Furthermore, the expression interpolates smoothly between these limits.

This approach has several advantages: (a) It incorporates essential physical ingredients by describing correctly the moments and multipoles of the charge distribution. (b) It provides a simple expression for the electric field, in the form of the ratio of two polynomials. (c) It is computationally fast. (d) It allows simplified self-consistent calculations, in which a few moments and multipole coefficients of the charge distribution evolve dynamically, rather than the distribution itself. A shortcoming of our method is that its simplicity is greatly diminished if one insists on representing accurately fields which do not vary smoothly. Also, its maximum simplicity (and therefore usefulness) is achieved in two-dimensional problems because in this case the field can be expressed very compactly in terms of complex variables.

Thus we assume in this note that we deal with infinitely long, straight beams whose charge density is time independent and depends only on the two transverse coordinates, $\rho = \rho(x, y)$, and we are concerned with the solution of Poisson's equation in free space (no attempt is made here to include the possible presence of conducting walls). This approximation is relevant to coasting beam problems, or to bunched beam problems in which the bunch length (in the lab frame) is much greater than the bunch width. It is a very good approximation for many accelerators, including the SSC and its injectors [5]. In fact, we have already applied the method described here to (non self-consistent) tracking simulations for the SSC's low-energy booster, for which the space-charge force has potentially the greatest effect [6].

Section 2 reviews the standard [7] long-distance and short-distance expansions of the electric field for arbitrary charge distribution in two dimensions. Section 3 presents the results of the expansions for a Gaussian distribution. Section 4 explains our method of approximation of functions in general. Section 5 applies the results of the previous section to represent the electric field of the Gaussian distribution to 5th order. Section 6 contains conclusions. The Appendix has a general discussion of the solution of Poisson's equation for any dimension $D$, and of the infrared divergence for $D = 2$.

# 2 Expansion of the Electric Field

Consider an infinitely long, straight, uniform charged beam moving with velocity $\mathbf{v}$. Then the Lorentz force on a co-moving charge $q$ is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, where all quantities are referred to the lab frame. It is straightforward to show from the Lorentz transformation that connects the beam's rest frame with the lab frame that $\mathbf{F} = q\mathbf{E}/\gamma^2$, where $\gamma$ is the usual relativistic factor. Therefore it is sufficient to calculate the electric field in order to compute the force.

The following discussion reviews the standard approximation methods [7] for the field. All quantities are referred to the lab frame. The field satisfies Gauss' law, $\nabla \cdot \mathbf{E} = 4\pi\rho(\mathbf{x})$, and we look for static solutions with free boundary conditions. We choose a gauge in which $\mathbf{E} = -\nabla\phi(\mathbf{x})$ where the potential is time-independent and satisfies

$$-\nabla^2\phi(\mathbf{x}) = 4\pi\rho(\mathbf{x}) \tag{2.1}$$

The equation for $\phi(\mathbf{x})$ is discussed in the Appendix, in which the "source" is $S(\mathbf{x}) = 4\pi\rho(\mathbf{x})$ (for MKS units replace $\rho(\mathbf{x}) \to \rho(\mathbf{x})/(4\pi\epsilon_0)$). The charge density $\rho(\mathbf{x})$ has dimensions of charge per unit volume, as usual, and is normalized so that

$$\int d^2\mathbf{x}\, \rho(\mathbf{x}) = \lambda \tag{2.2}$$

is the charge per unit length along the beam. The solution for the potential is

$$\phi(\mathbf{x}) = -2 \int d^2\mathbf{x}' \, \ln|\mathbf{x} - \mathbf{x}'|\, \rho(\mathbf{x}') \tag{2.3a}$$

$$= 4\pi \int \frac{d^2\mathbf{k}}{(2\pi)^2}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, \frac{\tilde{\rho}(\mathbf{k})}{\mathbf{k}^2} \tag{2.3b}$$

where $\tilde{\rho}(\mathbf{k})$ is the Fourier transform of $\rho(\mathbf{x})$,

$$\tilde{\rho}(\mathbf{k}) = \int d^2\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, \rho(\mathbf{x}) \tag{2.4}$$

In Eq. (2.3a) we have dropped the constant divergent term (see the Appendix for a full discussion of this infrared singularity) since it does not contribute to the electric field, but the infinity is still present in Eq. (2.3b). The corresponding expressions for $\mathbf{E}$ are

$$\mathbf{E}(\mathbf{x}) = 2 \int d^2\mathbf{x}' \, \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^2} \tag{2.5a}$$

$$= 4\pi \int \frac{d^2\mathbf{k}}{(2\pi)^2}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, (-i\mathbf{k}) \frac{\tilde{\rho}(\mathbf{k})}{\mathbf{k}^2} \tag{2.5b}$$

## 2.1 Point-Charge Case

If there is a point charge at the origin, (*i.e.*, the beam is a thin line), then $\rho(\mathbf{x}) = \lambda\, \delta^{(2)}(\mathbf{x})$ and $\tilde{\rho}(\mathbf{k}) = \lambda$. The potential (up to a constant) and the field are

$$\phi(\mathbf{x}) = -2\lambda \ln r, \qquad \mathbf{E}(\mathbf{x}) = 2\lambda\, \frac{\hat{\mathbf{x}}}{|\mathbf{x}|} \tag{2.6}$$

where $r = |\mathbf{x}|$ and $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$.

### 2.2 Round-Beam Case

If the distribution is round ($\rho(\mathbf{x}) = \rho(r)$) then the field is radial, and its magnitude may be obtained directly from Gauss' law,

$$\frac{1}{r}\frac{d}{dr}(rE(r)) = 4\pi\rho(r), \qquad E(r) = \frac{4\pi}{r}\int_0^r dr'\, r'\rho(r') \tag{2.7}$$

If there is a line charge at the origin, the above expression is still valid provided $\rho(r)$ is suitably defined with a delta function. The potential $\phi(r)$ can be obtained by integrating $E(r) = -d\phi(r)/dr$.

### 2.3 Short-Distance Expansion

We assume from now on that $\rho(\mathbf{x})$ is a sufficiently localized, *but not point-like*, distribution, and that the origin of coordinates is chosen at the center of this distribution. A multipole expansion in $\mathbf{k}$-space is obtained by expanding the exponential in Eq. (2.3b) at the origin,

$$\begin{aligned}\phi(\mathbf{x}) &= 4\pi\int\frac{d^2\mathbf{k}}{(2\pi)^2}\left(1 + i\mathbf{k}\cdot\mathbf{x} + \frac{(i\mathbf{k}\cdot\mathbf{x})^2}{2!} + \frac{(i\mathbf{k}\cdot\mathbf{x})^3}{3!} + \cdots\right)\frac{\tilde{\rho}(\mathbf{k})}{\mathbf{k}^2}\\ &= \phi^{(0)} + x_i\,\phi_i^{(1)} + x_i x_j\,\phi_{ij}^{(2)} + x_i x_j x_k\,\phi_{ijk}^{(3)} + \cdots\end{aligned} \tag{2.8}$$

with implied summation over repeated indices. The expansion coefficients are the completely symmetric $n$-th rank tensors

$$\phi_{ij\cdots}^{(n)} = \frac{4\pi\; i^n}{n!}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\,(k_i k_j\cdots)\frac{\tilde{\rho}(\mathbf{k})}{\mathbf{k}^2} \tag{2.9}$$

Note that the first term, $\phi^{(0)}$, is the infinite constant mentioned earlier. This is the only infinite term in the expansion except for the case of a line charge, where $\tilde{\rho}(\mathbf{k}) = \lambda$, in which case all terms are infinite. This is another manifestation of the infrared singularity in 2 dimensions. In this case it is not legitimate to expand the exponential inside the integral in Eq. (2.3b) (this should not be surprising because it is impossible to expand $\ln r$ in a power series at the origin).

For an even-parity charge distribution, where $\tilde{\rho}(-\mathbf{k}) = \tilde{\rho}(\mathbf{k})$, all odd-rank tensors in the expansion vanish, as should be the case in order to recover an expression with only even powers of $x_i$.

### 2.4 Long-Distance Expansion

This is the usual multipole expansion. It is obtained, in general, by expanding the Green function in a power series in $|\mathbf{x}'|/|\mathbf{x}|$. Thus we substitute

$$\begin{aligned}\ln|\mathbf{x}-\mathbf{x}'| = \ln r &- \frac{1}{r^2}\mathbf{x}\cdot\mathbf{x}' - \frac{1}{2r^4}\left(2(\mathbf{x}\cdot\mathbf{x}')^2 - \mathbf{x}^2\mathbf{x}'^2\right)\\ &- \frac{1}{3r^6}\left(4(\mathbf{x}\cdot\mathbf{x}')^3 - 3(\mathbf{x}\cdot\mathbf{x}')\mathbf{x}^2\mathbf{x}'^2\right)\\ &- \frac{1}{4r^8}\left(8(\mathbf{x}\cdot\mathbf{x}')^4 - 8(\mathbf{x}\cdot\mathbf{x}')^2\mathbf{x}^2\mathbf{x}'^2 + \mathbf{x}^4\mathbf{x}'^4\right) + \cdots\end{aligned} \tag{2.10}$$

in Eq. (2.3a), so that $\phi(\mathbf{x})$ is written

$$\phi(\mathbf{x}) = -2\lambda\ln r + \frac{x_i}{r^2}T_i^{(1)} + \frac{x_i x_j}{r^4}T_{ij}^{(2)} + \frac{x_i x_j x_k}{r^6}T_{ijk}^{(3)} + \frac{x_i x_j x_k x_l}{r^8}T_{ijkl}^{(4)} + \cdots \tag{2.11}$$

(repeated indices are summed over). The $T^{(n)}$'s are completely symmetric, traceless $n$-th rank tensors that span a basis for the two-dimensional rotation group. $T^{(1)}$ is the electric dipole moment of the distribution,

$T^{(2)}$ the electric quadrupole moment, etc. They are extracted from (2.3$a$), and the explicit forms are, after appropriate symmetrization,

$$T_i^{(1)} = 2 \int d^2\mathbf{x}\, \rho(\mathbf{x})\, x_i \tag{2.12a}$$

$$T_{ij}^{(2)} = \int d^2\mathbf{x}\, \rho(\mathbf{x}) \, \left[2\, x_i x_j - \mathbf{x}^2 \delta_{ij}\right] \tag{2.12b}$$

$$T_{ijk}^{(3)} = \frac{2}{3} \int d^2\mathbf{x}\, \rho(\mathbf{x}) \, \left[4\, x_i x_j x_k - \mathbf{x}^2 (\delta_{ij} x_k + 2 \text{ more})\right] \tag{2.12c}$$

$$T_{ijkl}^{(4)} = \frac{1}{6} \int d^2\mathbf{x}\, \rho(\mathbf{x}) \, \left[24\, x_i x_j x_k x_l - 4\, \mathbf{x}^2 (\delta_{ij} x_k x_l + 5 \text{ more}) + \mathbf{x}^4 (\delta_{ij}\delta_{kl} + 2 \text{ more})\right] \tag{2.12d}$$

where the "2 more" and "5 more" terms are similar to the ones displayed but with indices permuted in all possible ways so as to make each term in parenthesis completely symmetric. Note that an even-parity distribution causes all odd-rank tensors to vanish, so that the resulting expansion has only even powers of $x_i$, as it should.

It should be noted that, depending on the precise form of $\rho(\mathbf{x})$, the expansions (2.8)–(2.9) and (2.10)–(2.12) may be asymptotic (*i.e.*, they may have zero radius of convergence) because they arise from the interchange of integration with Taylor expansion for a singular integrand. Nevertheless, the first few terms are certainly valid approximations to the potential provided only that the integrals that define the $\phi^{(n)}$'s and $T^{(n)}$'s exist. Our method of approximating functions, which we present below, is very well suited to represent functions whose asymptotic expansions are known.

# 3 Gaussian Beam Distribution

In this case the distribution and its Fourier transform are

$$\rho(\mathbf{x}) = \lambda \, \frac{e^{-\frac{1}{2}\left(x^2/\sigma_x^2 + y^2/\sigma_y^2\right)}}{2\pi\sigma_x\sigma_y}, \qquad \tilde{\rho}(\mathbf{k}) = \lambda \, e^{-\frac{1}{2}(\sigma_x^2 k_x^2 + \sigma_y^2 k_y^2)} \tag{3.1}$$

and in fact there exists a closed-form expression for the electric field [1], namely

$$E_x - iE_y = \frac{-2i\sqrt{\pi}\,\lambda}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \left[ w\left( \frac{x + iy}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \right) - e^{-\frac{1}{2}\left(x^2/\sigma_x^2 + y^2/\sigma_y^2\right)}\, w\left( \frac{x\,\sigma_y/\sigma_x + iy\,\sigma_x/\sigma_y}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \right) \right] \tag{3.2}$$

where $w(z)$ is the complex error function [8]. Note that while this formula has apparent singularities in the round beam limit, the divergences cancel out when the limit is taken appropriately. In practical applications for protons, where the beam fluctuates in shape near the round configuration, the above formula may not be very useful because of the large cancellations between the terms (computational speed is not a problem because there are fast algorithms for the complex error function [3].

Since we want to illustrate our method for general charge distributions, we proceed to calculate the expansion of the potential and field with the methods developed in Section 2.

## 3.1 Round-Beam Case

In the round beam case, $\sigma_x = \sigma_y \equiv \sigma$, Eq. (2.7) is easily integrated to give the magnitude

$$|E| = \frac{2\,\lambda}{r} \left(1 - e^{-r^2/2\sigma^2}\right) \tag{3.3}$$

with cartesian components $E_x = (x/r)\,|E|$ and $E_y = (y/r)\,|E|$.

## 3.2 Long-Distance Expansion

To proceed with our approximation scheme for elliptical Gaussian beams, we calculate first the long-distance expansion. Note that since $\rho(\mathbf{x})$ is of even parity, all odd-$n$ tensors $T^{(n)}$ vanish. Furthermore, since the distribution is separable, that is,

$$\rho(\mathbf{x}) = \lambda\,\hat{\rho}_x(x)\hat{\rho}_y(y) \tag{3.4}$$

it is a little easier to integrate Eq. (2.3a) with the logarithm expanded according to Eq. (2.10) than to calculate the $T^{(n)}$'s themselves. All integrals reduce to the one-dimensional form

$$\int_{-\infty}^{\infty} dx\, x^{2n} \hat{\rho}_x(x) = \underbrace{1\cdot 3\cdot 5\cdots\ (2n-1)}_{n \text{ factors}}\, \sigma_x^{2n} \tag{3.5}$$

and $x \leftrightarrow y$. We have made the obvious definition

$$\hat{\rho}_x(x) = \frac{e^{-x^2/2\sigma_x^2}}{\sqrt{2\pi}\,\sigma_x} \tag{3.6}$$

and similarly for $y$. Note that the $\hat{\rho}$'s are normalized to unity. The resulting expression for the potential is then

$$\phi(\mathbf{x}) = -2\lambda\left[\log r - \frac{1}{2\,r^4}(x^2-y^2)(\sigma_x^2-\sigma_y^2) - \frac{3}{4\,r^8}\left((x^2-y^2)^2 - 4\,x^2y^2\right)(\sigma_x^2-\sigma_y^2)^2 + \cdots\right] \tag{3.7}$$

and $\mathbf{E}$ is obtained by taking the gradient. It turns out that the expressions simplify a great deal if they are expressed in complex form *via* the following definitions:

$$E \equiv E_x + iE_y, \quad z \equiv x+iy, \quad \xi \equiv \frac{x}{\sigma_x} + i\frac{y}{\sigma_y}, \quad \omega \equiv \frac{2(x+iy)}{\sigma_x+\sigma_y} \tag{3.8}$$

Then the long-distance expansion of the field is

$$E = \frac{2\,\lambda}{\bar{z}}\left[1 + (\sigma_x^2-\sigma_y^2)\frac{1}{\bar{z}^2} + 3\,(\sigma_x^2-\sigma_y^2)^2\frac{1}{\bar{z}^4} + \cdots\right] \tag{3.9}$$

(a bar denotes complex conjugation). Note that in the round-beam limit all terms beyond the first one vanish, as they should. Indeed, the exact result, Eq. (3.3), is written in this case as

$$E = \frac{2\,\lambda}{\bar{z}}\left[1 - e^{-|z|^2/2\sigma^2}\right] \tag{3.10}$$

and the exponential has an essential zero at infinity, so that its power series expansion in $1/|z|$ vanishes identically.

For future use in the next section, we write Eq. (3.9) in terms of $\omega$,

$$E = \left(\frac{2\,\lambda}{\sigma_x+\sigma_y}\right)\frac{2}{\bar{\omega}}\left[1 + \frac{4\,\alpha}{\bar{\omega}^2} + \frac{48\,\alpha^2}{\bar{\omega}^4} + \cdots\right] \tag{3.11}$$

where $\alpha$ is defined to be

$$\alpha \equiv \frac{\sigma_x-\sigma_y}{\sigma_x+\sigma_y} \tag{3.12}$$

(note that for a round beam $\alpha = 0$ and $\xi = \omega = z/\sigma$).

### 3.3 Short-Distance Expansion

We use here Eq. (2.3b) with the trick (A.6) and $\tilde{\rho}(\mathbf{k})$ given by (3.1),

$$\phi(\mathbf{x}) = 2\pi \int_0^\infty dt \int \frac{d^2\mathbf{k}}{(2\pi)^2}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, \tilde{\rho}(\mathbf{k})\, e^{-\frac{1}{2}t\mathbf{k}^2} = \lambda \int_0^\infty dt\, \frac{\exp\left[-\frac{1}{2}\left(\frac{x^2}{t+\sigma_x^2} + \frac{y^2}{t+\sigma_y^2}\right)\right]}{\sqrt{\left(t+\sigma_x^2\right)\left(t+\sigma_y^2\right)}} \tag{3.13}$$

Notice that the infrared singularity is present in the above integral, which diverges logarithmically as $t \to \infty$. A divergence occurs only for $D \leq 2$, since for higher dimensions the extra factors in the denominator make the integral convergent (the above expression generalizes in the obvious way to any $D$).

This formula is also a special case of the more general expression for an arbitrary elliptical charge distribution, $\rho(\mathbf{x}) = \rho(x^2/\sigma_x^2 + y^2/\sigma_y^2) \equiv \rho(u)$,

$$\phi(\mathbf{x}) = \pi\sigma_x\sigma_y \int_0^\infty \frac{dt}{\sqrt{\left(t+\sigma_x^2\right)\left(t+\sigma_y^2\right)}} \int_{U(t)}^\infty du\, \rho(u) \tag{3.14}$$

where $U(t) \equiv x^2/\left(t+\sigma_x^2\right) + y^2/\left(t+\sigma_y^2\right)$ [2, 9].

The infrared divergence in the potential (3.13) can be eliminated by formally subtracting the constant $\phi(0)$, thus

$$\phi(\mathbf{x}) - \phi(0) = \lambda \int_0^\infty dt\, \frac{\exp\left[-\frac{1}{2}\left(\frac{x^2}{t+\sigma_x^2} + \frac{y^2}{t+\sigma_y^2}\right)\right] - 1}{\sqrt{\left(t+\sigma_x^2\right)\left(t+\sigma_y^2\right)}} \tag{3.15}$$

is well defined. This subtraction does not affect the field.

The short-distance expansion is obtained from (3.13) by expanding the exponential (this is legitimate in spite of the divergence, which arises only from the first term in the expansion). The resulting integrals are of the form [1]

$$I_{m,n}(a,b) = \int_0^\infty dt\, \frac{1}{(t+a)^{m+1/2}\,(t+b)^{n+1/2}} \tag{3.16}$$

where $m, n = 0, 1, 2, \ldots$ ($I_{0,0}$ is the only divergent one). By using the relations

$$\frac{\partial I_{m,n}(a,b)}{\partial a} = -(m+\tfrac{1}{2}) I_{m+1,n}(a,b) \tag{3.17a}$$

$$\frac{\partial I_{m,n}(a,b)}{\partial b} = -(n+\tfrac{1}{2}) I_{m,n+1}(a,b) \tag{3.17b}$$

and the obvious symmetry $I_{m,n}(a,b) = I_{n,m}(b,a)$, one can calculate all the $I_{m,n}$'s by recursively differentiating

$$I_{1,0}(a,b) = \frac{2}{\sqrt{a}\,(\sqrt{a}+\sqrt{b})} \tag{3.18}$$

with respect to $a$ or $b$. To 4th order the result is (omitting the divergent term)

$$\phi(\mathbf{x}) - \phi(0) = \lambda\left[-\frac{1}{(\sigma_x+\sigma_y)}\left(\frac{x^2}{\sigma_x} + \frac{y^2}{\sigma_y}\right) + \frac{1}{12\,(\sigma_x+\sigma_y)^2}\left(\frac{2\,\sigma_x+\sigma_y}{\sigma_x^3}x^4 + \frac{6\,x^2y^2}{\sigma_x\sigma_y} + \frac{2\,\sigma_y+\sigma_x}{\sigma_y^3}y^4\right) + \cdots\right] \tag{3.19}$$

Since the electric field is the derivative of the potential, we lose one order from the above expansion. In practice, however, it is just as simple to compute the field by first taking the gradient of (3.13) and then expanding the exponential. We have done this to 5th order; in terms of $\xi$ and $\omega$, the result is the remarkably simple expression

$$E = \left(\frac{2\,\lambda}{\sigma_x + \sigma_y}\right)\xi\left[1 - \frac{\xi}{12}(\bar{\omega} + 2\,\bar{\xi}) + \frac{\xi^2}{240}\left(\bar{\omega}^2 + 3\,\bar{\omega}\bar{\xi} + 6\,\bar{\xi}^2\right) + \cdots\right] \tag{3.20}$$

In the round-beam limit we obtain

$$E = \frac{\lambda\,z}{\sigma^2}\left(1 - \frac{|z|^2}{4\,\sigma^2} + \frac{|z|^4}{24\,\sigma^4} + \cdots\right) \tag{3.21}$$

which is just the Taylor expansion of (3.10).

# 4 Approximation of Functions—General

We describe here how to represent functions whose analytic behavior and approximate shape are known, and whose Taylor series expansions at the origin and infinity are known to a finite order. The function is represented by an "approximant" which is the ratio of two polynomials, very similar to a Padé approximant. It joins smoothly the Taylor expansions at both ends so that its own Taylor expansion coincides with the given ones to the order specified. If the function considered is not smooth, the approximant is not a good representation and, in fact, may have singularities which the original function does not (see below).

We illustrate our method for odd functions first. Even functions are treated next. Functions without well-defined parity can be uniquely decomposed into even and odd parts, which can then be treated separately.

## 4.1 Odd-Parity Functions

Consider the simplest example of a function of a single real variable, $f(x)$, such that it satisfies the following: (1) it is an odd function, $f(-x) = -f(x)$; (2) it vanishes only at $x = 0$ and $x = \infty$; and (3) its power-series expansion is known to be

$$f(x) = \begin{cases} x + \mathcal{O}(x^3), & x \to 0 \\ \dfrac{2}{x} + \mathcal{O}(x^{-3}), & x \to \infty \end{cases} \tag{4.1}$$

Then an "approximant" $F(x)$ to this function is

$$F(x) = \frac{x}{1 + a\,x^2} \tag{4.2}$$

which has, by construction, the correct behavior at 0 and $\infty$. To determine $a$, Taylor-expand $F(x)$ at $\infty$, and obtain $a = 1/2$,

$$F(x) = \frac{x}{1 + x^2/2} \tag{4.3}$$

This method can be extended to functions of several variables, real or complex, analytic (in the sense of Cauchy) or not, and whose Taylor expansions at 0 and $\infty$ are known to any order. (If the order of the expansion at 0 is not the same as the order at $\infty$, the approximant exists but is not unique. Also, if the function vanishes at other points, or if it is not smooth, there are additional pitfalls. See the "Pitfalls and Subtleties" section below).

Consider now an odd function $f(x, y, z \ldots)$ which also vanishes only at the origin and at infinity, and whose expansions are

$$f(x, y, z \ldots) = \begin{cases} f_1 + f_3 + f_5 + \cdots & x, y, z \ldots \to 0 \\ f_{-1} + f_{-3} + f_{-5} + \cdots & x, y, z \ldots \to \infty \end{cases} \tag{4.4}$$

where the $f_n(x, y, z \ldots)$ are homogeneous functions of $x, y, z \ldots$ of degree $n$. Some arbitrary examples might be $f_5 = x^5 + 2y^3z^2$, $f_{-3} = 1/(z^3 + xyz)$, etc. Then the approximant is

$$F(x, y, z \ldots) = \frac{F_1 + F_3 + F_5}{1 + G_2 + G_4 + G_6} \tag{4.5}$$

where the $F_n$'s and $G_n$'s are homogeneous functions of $x, y, z \ldots$ of degree $n$, to be determined. This can be done by expanding $F$ at 0 and at $\infty$, and identifying the terms with the $f_n$'s. This method yields 6 nonlinear equations for the 6 unknowns $F_1, \ldots, G_6$. This system of equations almost always has a unique solution. Another method, which yields a completely equivalent set of *linear* equations, consists of multiplying Eq. (4.5) by the denominator of $F$, then replacing $F$ by $f$ and identifying terms of similar order. Thus we obtain from the expansion at the origin

$$F_1 + F_3 + F_5 = f_1 + (f_3 + f_1G_2) + (f_5 + f_3G_2 + f_1G_4) + \mathcal{O}(x^7) \tag{4.6}$$

and from the expansion at infinity,

$$F_5 + F_3 + F_1 = (f_{-1}G_6) + (f_{-1}G_4 + f_{-3}G_6) + (f_{-1}G_2 + f_{-3}G_4 + f_{-5}G_6) + \mathcal{O}(x^{-1}) \tag{4.7}$$

Note that the unknown terms in the expansions of $f(x, y, z \ldots)$, represented by $\cdots$ in Eq. (4.4), start to appear in the $\mathcal{O}(x^7)$ or $\mathcal{O}(x^{-1})$ terms. Consequently these cannot be used to supply equations.

The resulting inhomogeneous system of equations is

$$\begin{array}{ll} F_1 = f_1 & F_1 = f_{-1}G_2 + f_{-3}G_4 + f_{-5}G_6 \\ F_3 = f_3 + f_1G_2 & F_3 = f_{-1}G_4 + f_{-3}G_6 \\ F_5 = f_5 + f_3G_2 + f_1G_4 & F_5 = f_{-1}G_6 \end{array} \tag{4.8}$$

which is most conveniently solved numerically (the algebraic solution looks fairly complicated and is not illuminating).

## 4.2 Even-Parity Functions

Even functions are handled in the same way as odd functions. For example, suppose we have a function $f(x, y, z \ldots)$ that vanishes only at infinity, and whose expansions are

$$f(x, y, z \ldots) = \begin{cases} f_0 + f_2 + \cdots & x, y, z \ldots \to 0 \\ f_{-2} + f_{-4} + \cdots & x, y, z \ldots \to \infty \end{cases} \tag{4.9}$$

(If $f$ does not vanish as $x \to \infty$, work with $f(x, y, z \ldots) - f(\infty)$). The approximant is in this case

$$F(x, y, z \ldots) = \frac{F_0 + F_2}{1 + G_2 + G_4} \tag{4.10}$$

and the equations we need follow from identifying similar terms in

$$F_0 + F_2 = f_0 + (f_2 + f_0G_2) + \mathcal{O}(x^4) \tag{4.11a}$$

$$F_2 + F_0 = (f_{-2}G_4) + (f_{-2}G_2 + f_{-4}G_4) + \mathcal{O}(x^{-1}) \tag{4.11b}$$

## 4.3 Pitfalls and Subtleties

Sometimes the set of equations to be solved has no unique solution, or the solution is absurd, in the sense that it makes the approximant an unacceptable representation of the original function. These problems arise when some of the $f_n$'s are negative or unknown. We will not discuss all the problems and their acceptable solutions, but we will illustrate the generic cases with two simple examples.

First we show how perfectly harmless ambiguities may arise. Consider the even-function example above, in which the $f_2$ is not known (which is a *very* different situation from the case $f_2 = 0$). Thus

$$f(x, y, z \ldots) = \begin{cases} f_0 + \cdots & x, y, z \ldots \to 0 \\ f_{-2} + f_{-4} + \cdots & x, y, z \ldots \to \infty \end{cases} \tag{4.12}$$

We first try the simplest approximant that has the correct behavior at the origin and infinity,

$$F = \frac{F_0}{1 + G_2} \tag{4.13}$$

but soon realize that this does not work: we have an overdetermined set of 3 equations for 2 unknowns. While the above approximant is acceptable at the origin, its Taylor expansion at infinity, $F_0(1/G_2 - 1/G_2^2 + \cdots)$, would imply that $f_{-2}$ and $f_{-4}$ are related. Therefore we have the choice of either disregarding $f_{-4}$ (and contenting ourselves with an approximant to order $\mathcal{O}(x^{-2})$ at $\infty$), or using the next-order approximant,

$$F = \frac{F_0 + F_2}{1 + G_2 + G_4} \tag{4.14}$$

In this case, the analysis yields the underdetermined set of 3 equations for 4 unknowns

$$F_0 + F_2 = f_0 + \mathcal{O}(x^2) \tag{4.15a}$$

$$F_2 + F_0 = (f_{-2}G_4) + (f_{-2}G_2 + f_{-4}G_4) + \mathcal{O}(x^{-1}) \tag{4.15b}$$

The problem can certainly be solved, but not uniquely. It arises from the conflict between our ignorance of the function at the origin with our desire to represent it accurately at infinity. A choice has to be made. One simple possibility is to choose $G_2 = 0$ and solve for $F_2$ and $G_4$. Not all are acceptable: for example, the choice $G_4 = 0$ takes us back to the previous case. Also, any negative value for $G_4$ or a sufficiently large and negative choice for $G_2$ would introduce undesirable singularities in the approximant because the denominator will vanish for some values of $x$. In any case, whatever choice we make implies a specific behavior of $F(x)$ at the origin to order $\mathcal{O}(x^2)$.

Another problem arises when the original function has zeroes (or poles) somewhere between 0 and $\infty$. To illustrate this, consider the simplest example for the odd function discussed first. Assume that $f(x)$ vanishes, say, at $x = \pm 1$, but is otherwise a smooth function. Then its expansions might be

$$f(x) = \begin{cases} -x + \mathcal{O}(x^3), & x \to 0 \\ \dfrac{2}{x} + \mathcal{O}(x^{-3}), & x \to \infty \end{cases} \tag{4.16}$$

(note the necessary difference in sign between the two terms). Then the simplest approximant is found to be

$$F = \frac{-x}{1 - x^2/2} \tag{4.17}$$

which has the correct behavior at both ends, but has an unacceptable singularity at $x = \sqrt{2}$. The way to fix this is by using an approximant that is forced to vanish at $x = \pm 1$,

$$F = \frac{-x(1 - x^2)}{1 + a\,x^2 + b\,x^4} \tag{4.18}$$

Note that, because of the extra factor in the numerator, we are forced to enlarge the denominator to order $x^4$ in order to respect the behavior at $\infty$. We now have an underdetermined set of 1 equation for 2 unknowns, which yields $b = 1/2$ and leaves $a$ undefined. The singularity, however, is gone, and the approximant is an acceptable representation of the function. (The simplest choice is $a = 0$, but any other value greater than $-\sqrt{2}$ is acceptable).

From the above example it is clear how to find approximants for functions that are known to have zeroes or poles at specified locations: put in the appropriate factors in the numerator (for zeroes) and in the denominator (for poles). What remains should be smooth, and therefore representable by an approximant.

A far more subtle question is the converse: Suppose we know the Taylor expansions of a function at 0 and $\infty$, and we also know that the function vanishes only at these two points. Is there any guarantee that the approximant to this function will also not have zeroes or poles? This property is true in all the examples discussed in this note, but is there a general proof? And what about the cases in which the original function has cuts? These questions can probably be discussed within the framework of the theory of Padé approximants.

# 5 Approximation of the Electric Field

We now apply the above the method, to 5th order, to the expansion of the electric field of elliptical Gaussian beams. Since the field is, in this case, an odd function which vanishes only at the origin and at infinity, we use the first case discussed in the previous section. We define the complex function $F(\xi,\bar{\xi},\omega,\bar{\omega})$ by

$$E \equiv \left(\frac{2\,\lambda}{\sigma_x+\sigma_y}\right) F(\xi,\bar{\xi},\omega,\bar{\omega}) \tag{5.1}$$

and write a 5th order approximant for $F$,

$$F = \frac{F_1+F_3+F_5}{1+G_2+G_4+G_6} \tag{5.2}$$

From Eqs. (3.11) and (3.20) we read off the $f_n$'s,

$$\begin{aligned} f_1 &= \xi & f_{-1} &= \frac{2}{\bar{\omega}} \\ f_3 &= -\frac{\xi^2}{12}(\bar{\omega}+2\,\bar{\xi}) & f_{-3} &= \frac{8\,\alpha}{\bar{\omega}^3} \\ f_5 &= \frac{\xi^3}{240}(\bar{\omega}^2+3\,\bar{\omega}\bar{\xi}+6\,\bar{\xi}^2) & f_{-5} &= \frac{96\,\alpha^2}{\bar{\omega}^5} \end{aligned} \tag{5.3}$$

We will not exhibit the general solution to the system of equations (4.8) that determines the $F$'s and $G$'s because it is unnecesarily complicated and, in practical applications, it is better to solve it numerically by computer.

In the round-beam case, however, the solution is simple because $\xi=\omega=z/\sigma$ and also $f_{-3}=f_{-5}=0$. It is

$$F = \frac{\xi\left(1+\frac{1}{4}|\xi|^2+\frac{1}{24}|\xi|^4\right)}{1+\frac{1}{2}|\xi|^2+\frac{1}{8}|\xi|^4+\frac{1}{48}|\xi|^6} \tag{5.4}$$

It can be verified that the Taylor expansion at the origin yields Eq. (3.21) while the expansion at $\infty$ gives

$$F = \frac{2}{\bar{\xi}} + \mathcal{O}(\xi^{-7}) \tag{5.5}$$

*i.e.*, the $-3$ and $-5$ terms vanish, as they should.

Another particular case of interest is the 1st (dipole) order approximant for the elliptical beam. That is to say, we consider only $f_1$ and $f_{-1}$. This is similar to the simplest odd-function case of the previous section. The resulting expression is

$$F = \frac{\xi}{1+\frac{1}{2}\xi\bar{\omega}} \tag{5.6}$$

Table 1

$|F|$ *vs.* $r/\sigma$ for a round beam

| $r/\sigma$ | 1st-order | 5th-order | exact |
|---|---|---|---|
| 0.0 | 0.000 | 0.000 | 0.000 |
| 0.5 | 0.444 | 0.470 | 0.470 |
| 1.0 | 0.667 | 0.785 | 0.787 |
| 1.5 | 0.706 | 0.888 | 0.900 |
| 2.0 | 0.667 | 0.842 | 0.865 |
| 2.5 | 0.606 | 0.743 | 0.765 |
| 3.0 | 0.545 | 0.645 | 0.659 |
| 3.5 | 0.491 | 0.563 | 0.570 |
| 4.0 | 0.444 | 0.496 | 0.500 |
| 4.5 | 0.404 | 0.443 | 0.444 |
| 5.0 | 0.370 | 0.399 | 0.400 |
| 5.5 | 0.341 | 0.363 | 0.364 |
| 6.0 | 0.316 | 0.333 | 0.333 |
| 6.5 | 0.294 | 0.308 | 0.308 |
| 7.0 | 0.275 | 0.286 | 0.286 |
| 7.5 | 0.258 | 0.267 | 0.267 |
| 8.0 | 0.242 | 0.250 | 0.250 |
| 8.5 | 0.229 | 0.235 | 0.235 |
| 9.0 | 0.217 | 0.222 | 0.222 |
| 9.5 | 0.206 | 0.211 | 0.211 |
| 10.0 | 0.196 | 0.200 | 0.200 |

whose real part is

$$F_x = \frac{(\sigma_x + \sigma_y)^2 \left( \frac{x}{\sigma_x} + \frac{x}{\sigma_x + \sigma_y} \left( \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} \right) \right)}{\left( \sigma_x + \sigma_y + \frac{x^2}{\sigma_x} + \frac{y^2}{\sigma_y} \right)^2 + \left( \frac{xy}{\sigma_x \sigma_y} \right)^2 (\sigma_x - \sigma_y)^2} \tag{5.7}$$

(the imaginary part $F_y$ is most easily obtained by the replacements $x, \sigma_x \leftrightarrow y, \sigma_y$).

The crucial thing to notice in Eqs. (5.4) and (5.7) is that *neither the numerators nor the denominators vanish for physical values of $x$ and $y$.* This property is necessary in order that the approximant have physical meaning. It does not seem guaranteed by the algorithm that produces it. We have verified that this holds true to 5th order.

Another important property of Eq. (5.7) is seen by setting $y = 0$. If we use definition (5.1), the $x$-component of the electric field is

$$E_x(x, y = 0) = \frac{2\lambda\, x}{\sigma_x(\sigma_x + \sigma_y) + x^2} \tag{5.8}$$

In this case the crucial thing to notice is that *it has the correct asymptotic behavior as $x \to \infty$.* In deriving the expressions for the approximant we made the implicit assumption that all variables scale together to 0 or $\infty$. We see from this example that we get the right answer even if the variables do not scale together. We have verified that this also holds true to 5th order. Without this property the approximants would be of very limited use.

Finally, we provide a numerical comparison, for the round beam case, of the first-order approximant (Eq. (5.6)), the fifth-order one (Eq. (5.4)), and the exact result (Eq. (3.10)). The following table shows the absolute value of $F$, defined by Eq. (5.1), *vs.* radial distance in units of $\sigma$.

## 6 Discussion and Conclusions

We have presented an approximation scheme for rational representation of functions of several complex variables. The "approximant" is accurate at the origin and at infinity, and interpolates smoothly between these two limits. We have described how to apply this method to represent the electric field produced by an arbitrary charge density distribution, and applied it in detail, to 5th order, to elliptical Gaussian beams.

Some questions remain which we have not addressed, namely: (1) Why do the approximants have no zeroes or poles for physical values of the coordinates? In other words, why do they represent the analytic properties of the original function more faithfully than their algorithm seems to guarantee? How general is this property? (2) In the case of functions of more than one variable, why do the approximants have the correct asymptotic behavior even when the limits of the variables are not taken uniformly?

While these questions are of more general mathematical interest, they have no bearing on the application discussed here. The resulting expression for the electric field is simple, accurate, and fast to compute numerically, and therefore suitable for tracking applications [6].

In practice, of course, no charge distribution is exactly Gaussian, although this distribution is often used in many calculations. Our scheme provides a simple rational approximation for the field of this distribution which may be used as the starting point of numerical or analytical studies. For example, the 1st order approximant for the field, Eq. (5.6), is particularly appealing because of its simplicity.

The numerical comparison for the round-beam case shows that the 5th order approximant is probably acceptable for most applications, while the 1st order one is accurate only at short and long distances.

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## A Green Functions

It is well known that the inhomogeneous solution to Poisson's equation is singular in two dimensions for localized sources. The singularity is a logarithmic one, and appears in the expression for the potential but not the field. Therefore, in classical electrodynamics, this singularity is not more than an inconsequential nuisance, since only the fields have physical significance. In the quantum theory, however, this infrared singularity is responsible for the confinment of fermion pairs, and therefore has a dramatic effect on the spectrum (and dynamics) of the theory [10].

Since we deal with the potential for a substantial part of the calculation, and we may need the expressions for $D = 3$ (for short bunches applications, for example), we first discuss the solution to Poisson's equation for any spatial dimension $D \geq 2$, and specialize to $D = 2$ later on. This is a way to control the singularity, since it appears only in the limit $D = 2$ (the dimensionality of space can be thought of as a continuous regularization parameter [11].

Consider Poisson's equation for a function $\Psi(\mathbf{x})$,

$$-\nabla^2 \Psi(\mathbf{x}) = S(\mathbf{x}) \tag{A.1}$$

where $\mathbf{x}$ is a $D$-dimensional vector, and $S(\mathbf{x})$ is a localized "source." There are no boundary conditions at finite distance. The general solution is written in terms of the Green function $G(\mathbf{x} - \mathbf{x}')$ or its Fourier transform $\widetilde{G}(\mathbf{k})$ (the "propagator")

$$\Psi(\mathbf{x}) = \int d^D\mathbf{x}'\, G(\mathbf{x} - \mathbf{x}')S(\mathbf{x}') \tag{A.2a}$$

$$= \int \frac{d^D\mathbf{k}}{(2\pi)^D}\; e^{i\mathbf{k}\cdot\mathbf{x}}\, \widetilde{G}(\mathbf{k})\widetilde{S}(\mathbf{k}) \tag{A.2b}$$

($\widetilde{S}(\mathbf{k})$ is the Fourier transform of the source). Expression (A.2a) is more convenient for expansions at large $|\mathbf{x}|$, while (A.2b) is more convenient for small-$|\mathbf{x}|$ approximations.

The Green function satisfies

$$-\nabla^2_{\mathbf{x}}\, G(\mathbf{x}-\mathbf{x}') = \delta^{(D)}(\mathbf{x}-\mathbf{x}') \tag{A.3}$$

which is easily solved in $\mathbf{k}$-space yielding, for the inhomogeneous part,

$$\widetilde{G}(\mathbf{k}) = \frac{1}{\mathbf{k}^2} \tag{A.4}$$

or, in $\mathbf{x}$-space,

$$G(\mathbf{x}-\mathbf{x}') = \int \frac{d^D\mathbf{k}}{(2\pi)^D}\, \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{\mathbf{k}^2} \tag{A.5a}$$

$$= \frac{\Gamma(D/2-1)}{4\pi^{D/2}\;|\mathbf{x}-\mathbf{x}'|^{D-2}} \tag{A.5b}$$

In going from (A.5a) to (A.5b) we have employed the very useful identity [12]

$$\frac{1}{\mathbf{k}^2} = \frac{1}{2}\int_0^\infty dt\, e^{-\frac{1}{2}t\mathbf{k}^2} \tag{A.6}$$

and reversed the order of integration. The integral over $\mathbf{k}$ is then straightforward (it is the product of $D$ Fourier transform of Gaussians), and the integral over $t$ yields the gamma function. We also use this trick in the calculation of the field of a Gaussian distribution in Section 3.3.

For $D = 3$, Eq. (A.5b) yields the usual expression for the Coulomb potential,

$$G(\mathbf{x}-\mathbf{x}') = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|} \tag{A.7}$$

but when $D = 2$ it is evident that it has a singularity, which arises from the region of integration $\mathbf{k} \simeq 0$ in Eq. (A.5a). If we take the limit $D \to 2$ in (A.5b) we obtain

$$G(\mathbf{x}-\mathbf{x}') = -\frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{x}'| + \text{ infinite constant} \tag{A.8}$$

The "infinite constant" disappears upon taking the gradient, and therefore has no physical consequence. Note that the argument of the logarithm is not dimensionless. This can be corrected by replacing $|\mathbf{x}-\mathbf{x}'| \to |\mathbf{x}-\mathbf{x}'|/\ell$, where $\ell$ is an arbitrary length scale. This replacement is permissible because all it does is to add a constant to the potential, which has no effect on the electric field. Therefore we leave the argument as it stands.

The infinite constant can be given a physical interpretation by considering massive electrodynamics theory, in which the photon has a mass $m$. In this case, the appropriate equation for $\Psi$ is (in natural units),

$$(-\nabla^2 + m^2)\,\Psi(\mathbf{x}) = S(\mathbf{x}) \tag{A.9}$$

whose Green function can be calculated by the same methods. All we have to do is to replace $\mathbf{k}^2 \to \mathbf{k}^2 + m^2$ in the propagator. The integrals can be done for $D$ dimensions by using the same trick (A.6), and the result is

$$G(\mathbf{x}-\mathbf{x}') = \frac{1}{(2\pi)^{D/2}}\left(\frac{m}{|\mathbf{x}-\mathbf{x}'|}\right)^{D/2-1} K_{D/2-1}(m|\mathbf{x}-\mathbf{x}'|) \tag{A.10}$$

where $K_\nu(z)$ is a modified Bessel function [8].

For $D = 3$ this gives the Yukawa potential,

$$G(\mathbf{x} - \mathbf{x}') = \frac{e^{-m|\mathbf{x} - \mathbf{x}'|}}{4\pi|\mathbf{x} - \mathbf{x}'|} \tag{A.11}$$

and for $D = 2$ it is

$$G(\mathbf{x} - \mathbf{x}') = \frac{1}{2\pi} K_0(m|\mathbf{x} - \mathbf{x}'|) \tag{A.12}$$

which is perfectly regular. The infinity is not there because the photon mass prevents the propagator $\widetilde{G}(\mathbf{k})$ from diverging at $\mathbf{k} = 0$. If we now take the limit $m \to 0$ we obtain

$$G(\mathbf{x} - \mathbf{x}') = -\frac{1}{2\pi}(\ln|\mathbf{x} - \mathbf{x}'| + \ln m) + \text{constant} + \mathcal{O}(m^2) \tag{A.13}$$

which shows that the infinite constant in Eq. (A.8) is the logarithm of the photon mass.

# References


[1] M. Bassetti and G. A. Erskine, "Closed Expression for the Electric Field of a Two-dimensional Gaussian Charge," CERN-ISR-TH/80-06. See also the lectures by E. Keil and by G. H. Rees in "Theoretical Aspects of the Behaviour of Beams in Accelerators and Storage Rings," CERN 77-13.

[2] See, for example, E. Close, "Electric Fields for Beam-Beam Force Calculation: Subroutine DVXYLS," LBL-8903.

[3] See, for example, Y. Okamoto and R. Talman, "Rational Approximation of the Complex Error Function and the Electric Field of a Two-Dimensional Gaussian Charge Distribution," CBN 80-13, Sept. 1980.

[4] R. D. Ryne and A. Dragt, "Lie Algebraic Treatment of Space Charge," IEEE Accelerator Conference, 1987; R. D. Ryne, "Linear Transfer Maps for Beams with Space Charge," Univ. of Maryland Center for Theoretical Physics preprint.

[5] SSC Central Design Group, "Superconducting Supercollider Conceptual Design Report," SSC-SR-2020, March 1986.

[6] M. Furman, "Effect of the Space-Charge Force on Tracking at Low Energy," SSC-115.

[7] J. D. Jackson, "Classical Electrodynamics" (2nd edition), J. Wiley & Sons, 1975.

[8] M. Abramowitz and I. Stegun, *eds.*, "Handbook of Mathematical Functions," Dover Publications, 1965.

[9] R. L. Gluckstern, "Scalar Potential for Charge Distributions with Ellipsoidal Symmetry," Fermilab TM-1402.

[10] J. Schwinger, "Gauge Invariance and Mass. II," Phys. Rev. **128** (1962) 2425.

[11] G. 't Hooft and M. Veltman, "Regularization and Renormalization of Gauge Fields," Nucl. Phys. **B44** (1972) 189.

[12] J. Schwinger, "On Gauge Invariance and Vacuum Polarization," Phys. Rev. **82** (1951) 209.