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The Particle Density of a Gaussian Beam Matched to the Dynamical Beta Function

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Abstract

If the particle distribution in phase space is gaussian, its projection onto the nominal amplitude axis is not gaussian due to the "dynamical beta function effect." In this note we derive the appropriate formula, assuming that the beam is matched to the dynamical lattice functions.

1 Introduction.

When carrying out simulations (or measurements) of the particle density of a colliding beam, it is useful to know the limiting form when the beam-beam parameter is very small. In this limit, one can expect the integer and half-integer resonances to play the most significant role, and one can then calculate analytically the dependence of the density on the tune, assuming a certain functional form of the phase-space density.

In this limit, and to the extent that one is interested in the beam core, one can describe the beam-beam collision in linear approximation by a focusing kick whose strength is proportional to the beam-beam parameter ξ. As a result of this kick, the effective (or "dynamical") lattice functions α and β seen by the beam are different from the nominal ones and are functions of the tune and ξ. The beam is matched to the dynamical, rather than to the nominal, lattice functions

The problem we address here is not complicated, but it is not a tautology: even though we assume a certain functional form (gaussian) for the phase-space density, its projection onto the nominal amplitude axis is nontrivial because of the dynamical effect mentioned above. The complication arises from the mismatch between the dynamical functions affecting the beam and the nominal quantities used to calculate (or measure) the density.

2 Calculation.

Consider a gaussian beam with a large number N_0 of particles circulating in a linear, uncoupled, machine. The horizontal particle distribution at some observation point where there is no orbit distortion is given by

$$\rho(x,x') \equiv \frac{1}{N_0}\frac{dN}{dxdx'} = \frac{1}{2\pi\epsilon}\exp\left[-\frac{\sigma'^2x^2 - 2\chi xx' + \sigma^2x'^2}{2\epsilon^2}\right] \tag{1}$$

with a corresponding expression for the vertical counterpart. By definition, this distribution is normalized to unity, ie,

$$\int dxdx'\rho(x,x') = 1 \tag{2}$$

which implies the equality

$$\epsilon^2 = \sigma^2\sigma'^2 - \chi^2 \tag{3}$$

The average of any function $f(x,x')$ is given by

$$\langle f(x,x')\rangle = \int dxdx'\rho(x,x')f(x,x') \tag{4}$$

In particular, σ, σ' and χ are the averages

$$\sigma^2 = \langle x^2\rangle, \qquad \sigma'^2 = \langle x'^2\rangle, \qquad \chi = \langle xx'\rangle \tag{5}$$

The density is such that the linear averages $\langle x\rangle$ and $\langle x'\rangle$ vanish, in accordance with the assumption that there is no closed orbit distortion at the observation point. The quantity ϵ is the beam emittance, and it is given by the usual expressions, Eqs. (3-5).

Now if the lattice functions at the observation point are α, β and γ (with $\beta\gamma = 1+\alpha^2$), we say that the beam is matched to the lattice if it satisfies the conditions

$$\sigma^2 = \epsilon\beta, \qquad \sigma'^2 = \epsilon\gamma, \qquad \chi = -\epsilon\alpha \tag{6}$$

in which case the density (1) reads

$$\rho(x,x') = \frac{1}{2\pi\epsilon}\exp\left[-\frac{\gamma x^2 + 2\alpha xx' + \beta x'^2}{2\epsilon}\right] \equiv \frac{1}{2\pi\epsilon}\exp\left[-\frac{1}{2\epsilon}(x,x')\,M\begin{pmatrix} x \\ x' \end{pmatrix}\right] \tag{7}$$

where M is the matrix

$$M = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \tag{8}$$

Assume now that a linear kick is added to the lattice. This kick modifies the lattice functions and the tune of the ring. In the case when the kick is caused by a beam-beam collision, the modified lattice functions are referred to as "dynamical," and we call them α', β' and γ' (with $\beta'\gamma' = 1+\alpha'^2$). If our observation point is immediately before the collision point, the dynamical functions are given by the matrix equation

$$\begin{pmatrix} C'+\alpha'S' & \beta'S' \\ -\gamma'S' & C'-\alpha'S' \end{pmatrix} = \begin{pmatrix} C+\alpha S & \beta S \\ -\gamma S & C-\alpha S \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -4\pi\xi/\beta & 1 \end{pmatrix} \tag{9}$$

where $C \equiv \cos(2\pi\nu)$, $C' \equiv \cos(2\pi\nu')$, the S's are the corresponding sine functions, ν is the tune, and ξ is the strength of the beam-beam kick. The standard solution for this equation is [1]

$$C' = C - 2\pi\xi S \tag{10a}$$

$$\alpha' = \alpha - 2\pi\xi\frac{\beta'}{\beta} \tag{10b}$$

$$\frac{\beta'}{\beta} = \frac{S}{S'} = \frac{1}{\sqrt{1+4\pi\xi\cot(2\pi\nu)-(2\pi\xi)^2}} \tag{10c}$$

As a result of this beam-beam kick, the beam becomes matched to the dynamical lattice functions (we assume that the emittance does not change). Thus the dynamical matching conditions are

$$(\sigma^2)' = \epsilon\beta', \qquad (\sigma'^2)' = \epsilon\gamma', \qquad (\chi)' = -\epsilon\alpha' \tag{11}$$

and the new distribution is

$$\rho'(x,x') = \frac{1}{2\pi\epsilon}\exp\left[-\frac{1}{2\epsilon}(x,x')\,M'\begin{pmatrix} x \\ x' \end{pmatrix}\right], \qquad M' = \begin{pmatrix} \gamma' & \alpha' \\ \alpha' & \beta' \end{pmatrix} \tag{12}$$

Let us now define the normalized coordinates q and p via

$$\begin{pmatrix} x \\ x' \end{pmatrix} = U\begin{pmatrix} q \\ p \end{pmatrix}, \qquad U \equiv \frac{1}{\sqrt{\beta}}\begin{pmatrix} \beta & 0 \\ -\alpha & 1 \end{pmatrix} \tag{13}$$

The important thing to notice here is that it is the *nominal* lattice functions that enter the definition q and p, not the dynamical ones. The distribution (12) then reads

$$\frac{1}{N_0}\frac{dN'}{dqdp} = \frac{1}{2\pi\epsilon}\exp\left[-\frac{1}{2\epsilon}(q,p)\,B\begin{pmatrix} q \\ p \end{pmatrix}\right] \tag{14}$$

where B is the matrix $B = U^T M' U$ and we have used $dxdx' = dqdp$.

We now go to action-angle variables A and ϕ via the transformation

$$q = A\cos\phi \tag{15a}$$

$$p = -A\sin\phi \tag{15b}$$

and we define the dimensionless "normalized amplitude" $\hat{A}$ via

$$\hat{A}^2 \equiv \frac{A^2}{\epsilon} = \frac{q^2+p^2}{\epsilon} = \frac{x^2+(\beta x'+\alpha x)^2}{\sigma^2} \tag{16}$$

Notice, again, that it is the nominal, not dynamical, quantities that appear here. In terms of these variables, the dynamical density (14) reads

$$\frac{1}{N_0}\frac{dN'}{d\hat{A}^2 d\phi} = \frac{1}{4\pi}\exp\left[-\frac{\hat{A}^2}{2}\left(B_{11}\cos^2\phi + B_{22}\sin^2\phi - 2B_{12}\sin\phi\cos\phi\right)\right] \tag{17}$$

where we have used $dqdp = (\epsilon/2)d\hat{A}^2 d\phi$. If we are interested only in the amplitude distribution, we integrate over ϕ and obtain

$$\frac{1}{N_0}\frac{dN'}{d\hat{A}^2} = \int_0^{2\pi} d\phi\,\frac{1}{N_0}\frac{dN'}{d\hat{A}^2 d\phi} = \frac{1}{2}\exp\left[-\frac{\hat{A}^2}{4}\left(B_+ + B_-\right)\right] I_0\left[\frac{\hat{A}^2}{4}\left(B_+ - B_-\right)\right] \tag{18}$$

where I_0 is the usual modified Bessel function, and the $B_\pm$ are the eigenvalues of the matrix B. These are given by

$$B_\pm = \frac{1}{2}\left(T \pm \sqrt{T^2-4D}\right), \qquad \begin{cases} T \equiv \mathrm{tr}B = \beta\gamma' + \gamma\beta' - 2\alpha\alpha' \\ D \equiv \det B = 1 \end{cases} \tag{19}$$

Eq. (18), along with definitions (19), is our main result. In the limit when $\xi = 0$, B is the unit matrix, and Eq. (18) reduces, as it should, to a simple gaussian,

$$\frac{1}{N_0}\frac{dN'}{d\hat{A}^2} = \frac{1}{2}e^{-\hat{A}^2/2} \tag{20}$$

Finally, we note that a simplification occurs if $\alpha = 0$ at the collision point, as is normally the case. In this case we obtain, with a bit of straightforward algebra,

$$T = 2r\left(1 + 2\pi\xi\cot(2\pi\nu)\right) \tag{21a}$$

$$\sqrt{T^2-4D} = \frac{4\pi\xi r}{|\sin(2\pi\nu)|} \tag{21b}$$

where we have defined $r \equiv \beta'/\beta$, which is given by Eq. (10c). This completes our calculation.

A more conventional way to express the dynamical density is obtained from Eq. (12) by integrating out x'. If we define the normalized coordinate $\hat{x} \equiv x/\sigma$, we obtain

$$\frac{1}{N_0}\frac{dN'}{d\hat{x}} = \frac{e^{-\hat{x}^2/2r}}{\sqrt{2\pi r}} \tag{22}$$

Figure 1 shows the densities $(dN'/d\hat{A}^2)/N_0$, Eq. (18), and $(dN'/d\hat{x})/N_0$, Eq. (22), for two selected values of the tune. Obviously, for $\xi = 0$, one obtains a gaussian density in either case (note, however, that the normalization is different in both cases). The conventional density is more sensitive to the tune and to ξ than the normalized amplitude density, particularly near the beam center. This is because the slope of this latter as a function of $\hat{A}^2/2$ is proportional to $T/2 = 1 + \mathcal{O}(\xi^2)$, while the slope of the conventional density as a function of $\hat{x}^2/2$ is proportional to $r^{-1} = 1 + \mathcal{O}(\xi)$.

Figure 2 shows the results of a weak-strong multiparticle simulation [2] for $\nu_x = 0.57$, $\nu_y = 0.64$ and $\xi = 0.03$ along with the analytic results derived above. The simulation parameters correspond to those of PEP-II [3] and include long-bunch effects and synchrotron oscillations. The density is that of the weak (low-energy) beam; it is obtained by tracking 1024 representative particles for 5000 turns, ie., approximately one damping time. The ring lattice is represented by a decoupled 4×4 linear map. The amplitudes and coordinates of all particles are recorded at every turn, and then binned into a histogram in $\hat{A}^2$ or $\hat{x}^2$, which is plotted in the figure.

References

[1] See, for example, M. A. Furman, "Dynamical Beta Function and Beam-Beam Tune Shift for PEP-II," ESG Tech Note 232/PEP-II/AP Note 4-93, January 3rd, 1994; "Beam-Beam Tune Shift and Dynamical Beta Function in PEP-II," LBL-34957/CBP Note-057/PEP-II AP Note 13-94, June 1994.

[2] M. A. Furman, A. Zholents, T. Chen and D. Shatilov, "Comparison of Beam-Beam Code Simulations," CBP Tech Note-59, PEP-II/AP Note 95-04, July 13, 1995; "Comparisons of Beam-Beam Simulations," LBL-37680/CBP Note-152/PEP-II AP Note 95.39, Proc. 7th Advanced ICFA Workshop on Beam Dynamics, JINR, Dubna, Russia, 18-20 May, 1995, pp. 123-142.

[3] "PEP-II: An Asymmetric B Factory–Conceptual Design Report," June 1993, LBL-PUB-5379/SLAC-418/CALT-68-1869/UCRL-ID-114055/UC-IIRPA-93-01.

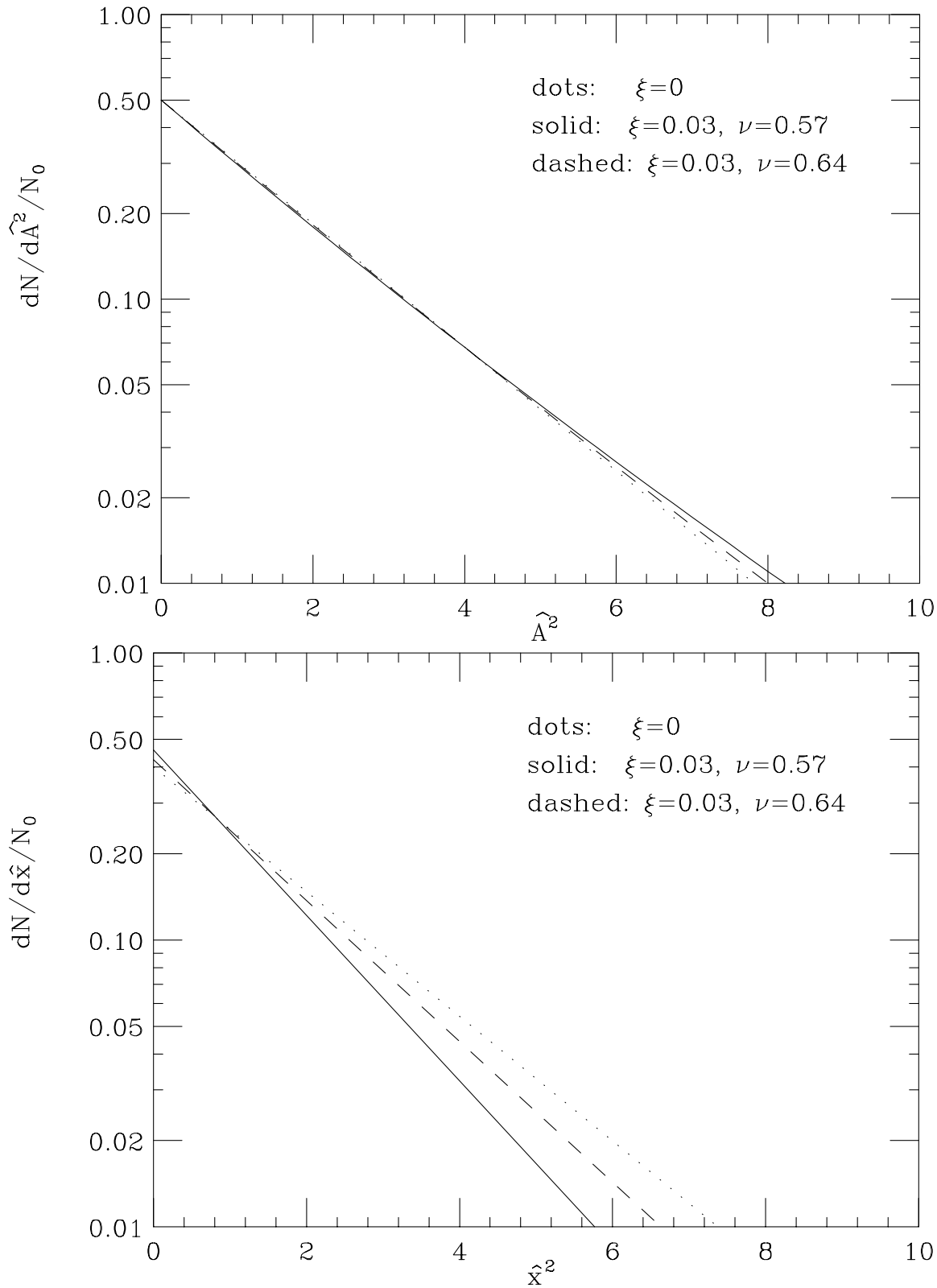

Figure 1: Top: the density $(dN'/d\hat{A}^2)/N_0$, Eq. (18), for two values of the tune. Bottom: the density $(dN'/d\hat{x})/N_0$, Eq. (22).

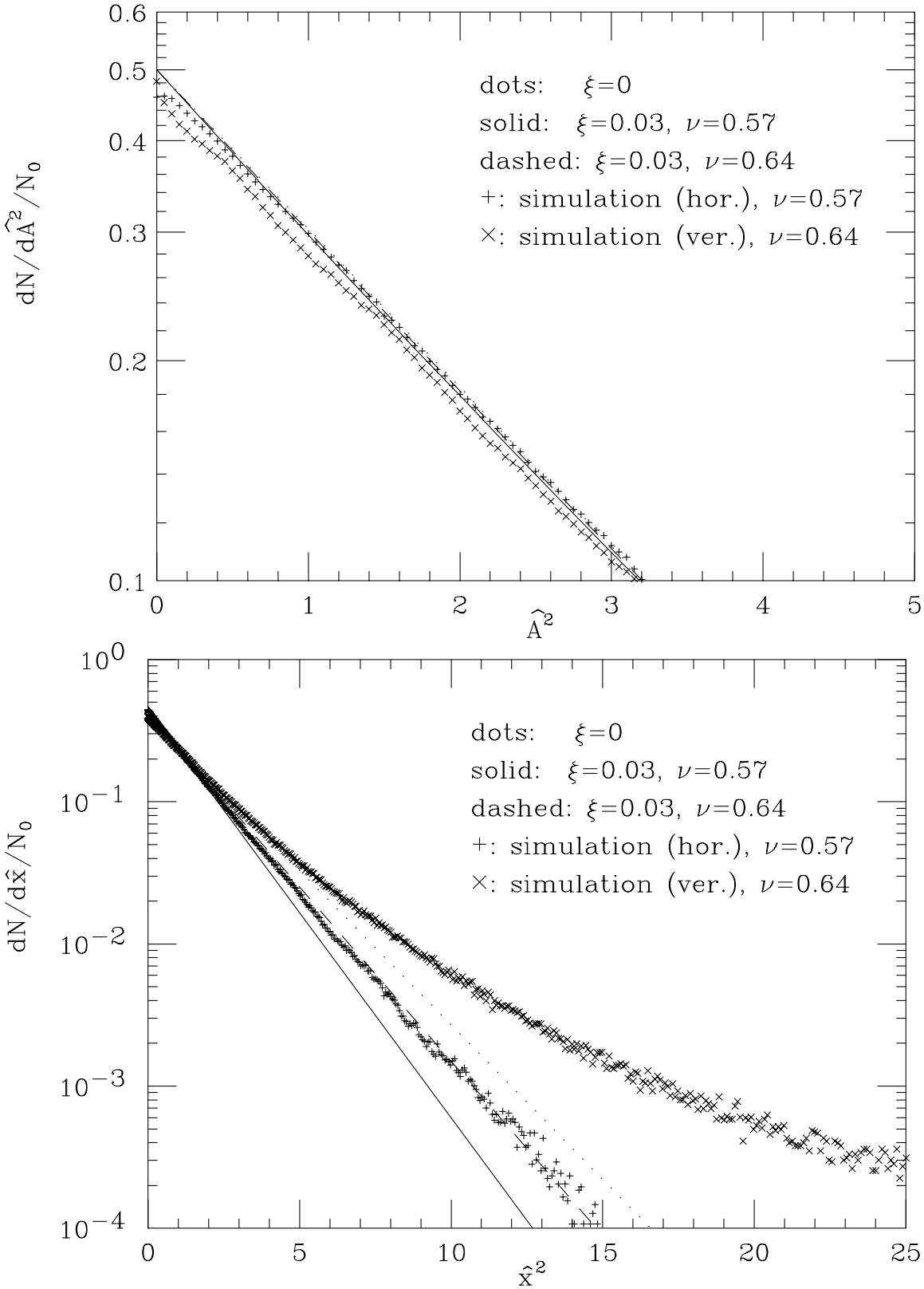

Figure 2: Simulation results along with the same analytic results shown in Fig. 1.