Practical Formula for Bunch Power Loss in Resonators of Almost Arbitrary Quality Factor

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Abstract

We derive an approximate formula, based on the complex error function, for the power lost by a gaussian bunch in periodic orbit traversing a resonator. We state the conditions that the bunch length σ_z, quality factor Q and resonant frequency ω_r must satisfy in order that this formula be valid.

1 Introduction

Consider a charged particle bunch moving in a periodic orbit of length $2\pi R$ with frequency f_0. If this bunch traverses a resonant structure with impedance

$$Z(\omega) = \frac{R_S}{1 + iQ\left(\dfrac{\omega_r}{\omega} - \dfrac{\omega}{\omega_r}\right)} \tag{1}$$

then the power loss is [1]

$$P = (cf_0)^2 \sum_{m=-\infty}^{\infty} |\tilde{\rho}(m\omega_0)|^2 \,\mathrm{Re}\, Z(m\omega_0) \tag{2}$$

where $\tilde{\rho}(\omega)$ is the frequency spectrum of the longitudinal charge density $\rho(z)$,

$$\tilde{\rho}(\omega) = \frac{1}{c} \int_{-\pi R}^{\pi R} dz\, e^{i\omega z/c} \rho(z) \tag{3}$$

If $\rho(z)$ varies smoothly and is nonzero over a distance comparable to $2\pi R$, then $\tilde{\rho}(\omega)$ is significantly different from zero over a small region of ω (measured in units of $\omega_0 \equiv 2\pi f_0$), and then a few terms in the summation in Eq. (2) yield an accurate estimate for the power loss.

If, on the other hand, $\tilde{\rho}(\omega)$ is very broad-banded, it is necessary to keep a large number of terms in the summation in order to achieve good accuracy, and therefore a better method is desirable. This case arises when $\rho(z)$ is nonzero over a very small region of z, that is, when the bunch is much shorter than the length of the orbit. This is clearly the case for large circular accelerators such as the SSC, where the circumference is millions of times greater than the bunch length, and therefore an accurate evaluation of the power loss may require millions of terms in Eq. (2). This is the limiting case we address here.

2 Derivation

We assume, therefore, that $\tilde{\rho}(\omega)$ varies little over a frequency interval of size ω_0. In order to find a useful approximation for Eq. (2) we assume also that $\mathrm{Re}\, Z(\omega)$ varies smoothly over such an interval. In this case it is legitimate to replace the summation by an integral in Eq. (2),

$$P = (cf_0)^2 \int_{-\infty}^{\infty} \frac{d\omega}{\omega_0} |\tilde{\rho}(\omega)|^2 \,\mathrm{Re}\, Z(\omega) \tag{4}$$

which is easier to evaluate accurately.

The condition of smooth variation of $\mathrm{Re}\, Z(\omega)$ is easy to state more precisely. Eq. (1) implies

$$\mathrm{Re}\, Z(\omega) = \frac{R_S}{1 + Q^2\left(\dfrac{\omega_r}{\omega} - \dfrac{\omega}{\omega_r}\right)^2} \tag{5}$$

so the fastest variation occurs around the resonant peaks at $\omega = \pm\omega_r$. The FWHM of these peaks is $\Delta\omega = \omega_r/Q$ and therefore the smooth-variation condition of $\mathrm{Re}\, Z(\omega)$ translates into the requirement

$$\frac{\omega_r}{Q} \gg \omega_0 \tag{6}$$

We now consider a bunch of total charge Ne with gaussian longitudinal charge density,

$$\rho(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2}\right) \tag{7}$$

whose frequency spectrum is[1]

$$\tilde{\rho}(\omega) = \frac{Ne}{c} \exp\left(-\tfrac{1}{2}\omega^2 {\sigma_t}^2\right) \tag{8}$$

[1] We take the liberty to extend to infinity the limits of integration in Eq. (3) in anticipation of our approximation.

where $\sigma_t \equiv \sigma_z/c$. The smooth-variation condition described above translates into the requirement $(\omega_0\sigma_t)^2 \ll 1$ or

$$\left(\frac{\sigma_z}{R}\right)^2 \ll 1 \tag{9}$$

which is usually well satisfied in high-energy storage rings. From here on we assume the validity of inequalities (6) and (9).

An obvious change of variable in Eq. (4) yields

$$P = R_S I_b^2 \left(\frac{R}{\sigma_z}\right) K(\alpha, Q) \tag{10}$$

where $I_b \equiv Nef_0$ is the bunch current, $\alpha = \omega_r\sigma_t$ and $K(\alpha, Q)$ is a function defined by the integral

$$K(\alpha, Q) \equiv \int_{-\infty}^{\infty} ds \, \frac{e^{-s^2}}{1 + Q^2\left(\frac{s}{\alpha} - \frac{\alpha}{s}\right)^2} \tag{11}$$

We consider now the following representation of the complex error function [2]

$$w(z) = \frac{iz}{\pi}\int_{-\infty}^{\infty} ds \, \frac{e^{-s^2}}{z^2 - s^2} \tag{12}$$

valid for $\mathrm{Im}\, z > 0$, and calculate

$$\mathrm{Re}(zw(z)) = \frac{1}{\pi}\int_{-\infty}^{\infty} ds \, e^{-s^2}\, \mathrm{Re}\left(\frac{iz^2}{z^2 - s^2}\right) \tag{13}$$

By setting $z = x + iy$ we obtain

$$\mathrm{Re}\left(\frac{iz^2}{z^2 - s^2}\right) = \frac{2xys^2}{(x^2+y^2)^2 + s^4 - 2(x^2-y^2)s^2} \tag{14}$$

while the integrand in Eq. (11) is proportional to

$$\frac{s^2}{\alpha^4 + s^4 - 2(1 - 1/2Q^2)\alpha^2 s^2} \tag{15}$$

Therefore we are led to identify

$$x^2 + y^2 = \alpha^2 \tag{16a}$$
$$x^2 - y^2 = \alpha^2(1 - 1/2Q^2) \tag{16b}$$

from which we obtain

$$x = \frac{\alpha}{2Q}\sqrt{4Q^2 - 1} \tag{17a}$$

$$y = \frac{\alpha}{2Q} \tag{17b}$$

(the other solutions are not appropriate). Therefore

$$K(\alpha, Q) = \frac{2\pi\, \mathrm{Re}(zw(z))}{\sqrt{4Q^2 - 1}} \tag{18}$$

and, finally,

$$P = R_S I_b^2 \left(\frac{2\pi R}{\sigma_z}\right) \frac{\mathrm{Re}(zw(z))}{\sqrt{4Q^2 - 1}} \tag{19}$$

where

$$z = \frac{\alpha}{2Q}\left(\sqrt{4Q^2 - 1} + i\right) \tag{20}$$

Eqs. (19–20) constitute our final result, which is valid provided the inequalities (6) and (9) are satisfied. The virtue of (19) lies in the fact that the complex error function can be easily estimated numerically by efficient routines available commercially.

3 Remarks

1. For $Q = 1$ Eq. (19) yields the known result [3].
2. Despite its appearance, Eq. (19) is not divergent at $Q = 1/2$ because there z is purely imaginary and $w(z)$ is purely real, hence $\mathrm{Re}(zw(z)) = 0$. For $Q < 1/2$ Eq. (19) is clearly not valid, although a generalization is probably easy to find.
3. For $Q \gg 1$ but not so large that (6) is violated, z becomes efectively real hence [2] $\mathrm{Re}(zw(z)) \simeq x\exp(-x^2)$ and (19) simplifies to

$$P \simeq R_S I_b^2 \left(\frac{\pi\omega_r}{\omega_0 Q}\right) e^{-(\omega_r\sigma_t)^2} \tag{21}$$

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References

[1] A. W. Chao, in *Physics of High Energy Particle Accelerators*, M. Month, ed., AIP Conf. Proc. **105**, p. 353 (SLAC Summer School, 1982).

[2] *Handbook of Mathematical Functions*, M. Abramowitz and I. A. Stegun, eds., Dover, 1965, p. 297.

[3] Reference Designs Study for the SSC, p. 138.