# Simple Method to Symplectify Matrices

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## I. Introduction

For matrices which are "approximately symplectic", in a sense defined below, we present a simple algorithm to render them symplectic. The algorithm is a rapidly convergent procedure which only involves a few matrix multiplies.

## II. Method

Consider a (2n) x (2n) matrix M belonging to the symplectic group Sp(2n). By definition, it satisfies the constraint

$$M J \tilde{M} = J \tag{1}$$

where J is the matrix

$$J = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right)$$

and $\tilde{M}$ is the transpose of M. Alternatively, eq. (1) can be written in the form

$$-M J \tilde{M} J = 1 \tag{2}$$

We define an "approximately symplectic" matrix M one that satisfies

$$-MJ\tilde{M}J = 1 + E \tag{3}$$

where E is a (2n) x (2n) matrix which is "small" in a sense that remains to be specified. An exactly symplectic matrix M', which is close to M, can be constructed by multiplying M by a correction matrix C

$$M' = CM \tag{4}$$

and imposing the symplectic condition, eq. (1), on M'. In terms of M and C it reads

$$-CMJ\tilde{M}\tilde{C}J = 1 \tag{5}$$

However, eq. (3) implies $MJ\tilde{M} = (1+E)J$, so we find

$$-C(1+E)J\tilde{C}J = 1 \tag{6}$$

Of course, eq. (6) has an infinite number of solutions for C, corresponding to the infinite number of ways of making a nonsymplectic matrix symplectic. We require, however, that C reduces to the identity when E = 0. A simple solution for C which satisfies this property is obtained by imposing the additional <u>assumption</u>

$$J\tilde{C}J = -C \tag{7}$$

so we easily find from (6)

$$C = (1+E)^{-\frac{1}{2}} \tag{8}$$

A necessary condition for C to exist is that all eigenvalues of E be $\neq$ $-1$. In practical applications, however, all elements of E are small (in absolute value) compared with unity; if they are sufficiently small, which we assume to be the case, then we have a sufficient condition for the existence of C.

It is straightforward to check that C satisfies assumption (7): First, it is evident that eq. (7) is satisfied by the unit matrix, i.e., when C = 1. Second, it is simple to verify that (7) is also satisfied by the matrix E, as defined by eq. (3). This completes a formal proof. A more concrete proof consists of expanding C in a power series in E, assumed to converge, and proving by induction that (7) is satisfied for any power of E.

## III. Practical procedure

In practice the computation of $(1+E)^{-\frac{1}{2}}$ is too time-consuming for this method to be competitive with other, more conventional ones.[1] However, if E is sufficiently small, our method leads to a simple and rapidly convergent iterative procedure, by approximating

$$C \simeq 1 - \tfrac{1}{2}E \tag{9}$$

Thus the improved matrix M' defined by

$$\begin{aligned} M' &= \left(1 - \tfrac{1}{2}E\right)M \\ &= \tfrac{1}{2}\left(3 + MJ\tilde{M}J\right)M \end{aligned} \tag{10}$$

deviates from symplecticity by terms of order $E^2$. A second iteration yields a deviation of order $E^4$, a third of order $E^8$, and so on.

It appears at first sight that the computation of M', eq. (10), requires 4 matrix multiplies and one matrix addition. This is deceiving: if we break up M into four n x n block matrices,

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{11}$$

then we easily find

$$MJ\tilde{M}J = \left(\begin{array}{c|c} B\tilde{C} - A\tilde{D} & A\tilde{B} - \widetilde{(A\tilde{B})} \\ \hline D\tilde{C} - \widetilde{(D\tilde{C})} & \widetilde{(B\tilde{C} - A\tilde{D})} \end{array}\right) \tag{12}$$

which requires the computation of the n x n matrix products $B\tilde{C}$, $A\tilde{D}$, and only n(n-1)/2 elements of the products $A\tilde{B}$ and $D\tilde{C}$.

For matrices M whose deviation from symplecticity is a priori unknown, it seems appropriate to monitor the convergence of the procedure by computing a parameter that measures the magnitude of E. One possibility is the determinant of E (or the determinant of M). A more pessimistic and reliable estimate (but also more time-consuming) is the RMS of the matrix elements of E,

$$\mathcal{E} = \sqrt{\frac{1}{N} \sum_{ij} E_{ij}^2} \tag{13}$$

where $N=4n^2$ is the number of elements. Of course one may use a simpler and faster estimator if E is known to be close to zero, or dispense with this monitoring altogether.

## IV. Conclusion

We have described a simple and practical iterative method to render an approximately symplectic matrix symplectic. If the matrix is not too far from the group, the procedure converges exponentially with the number of iterations. The relative efficiency of the method is greater the larger is the dimensionality of the matrix. A numerical study of this method, including applications to tracking calculations, and a comparison with other methods will be presented elsewhere.[2]

## V. References

1. H. Wiedemann, PEP-220, Sept. 1976; D. Douglas and R.V. Servranckx, LBL-SSC Note 28.

2. D. Douglas, A. Dragt and M. Furman, in preparation.