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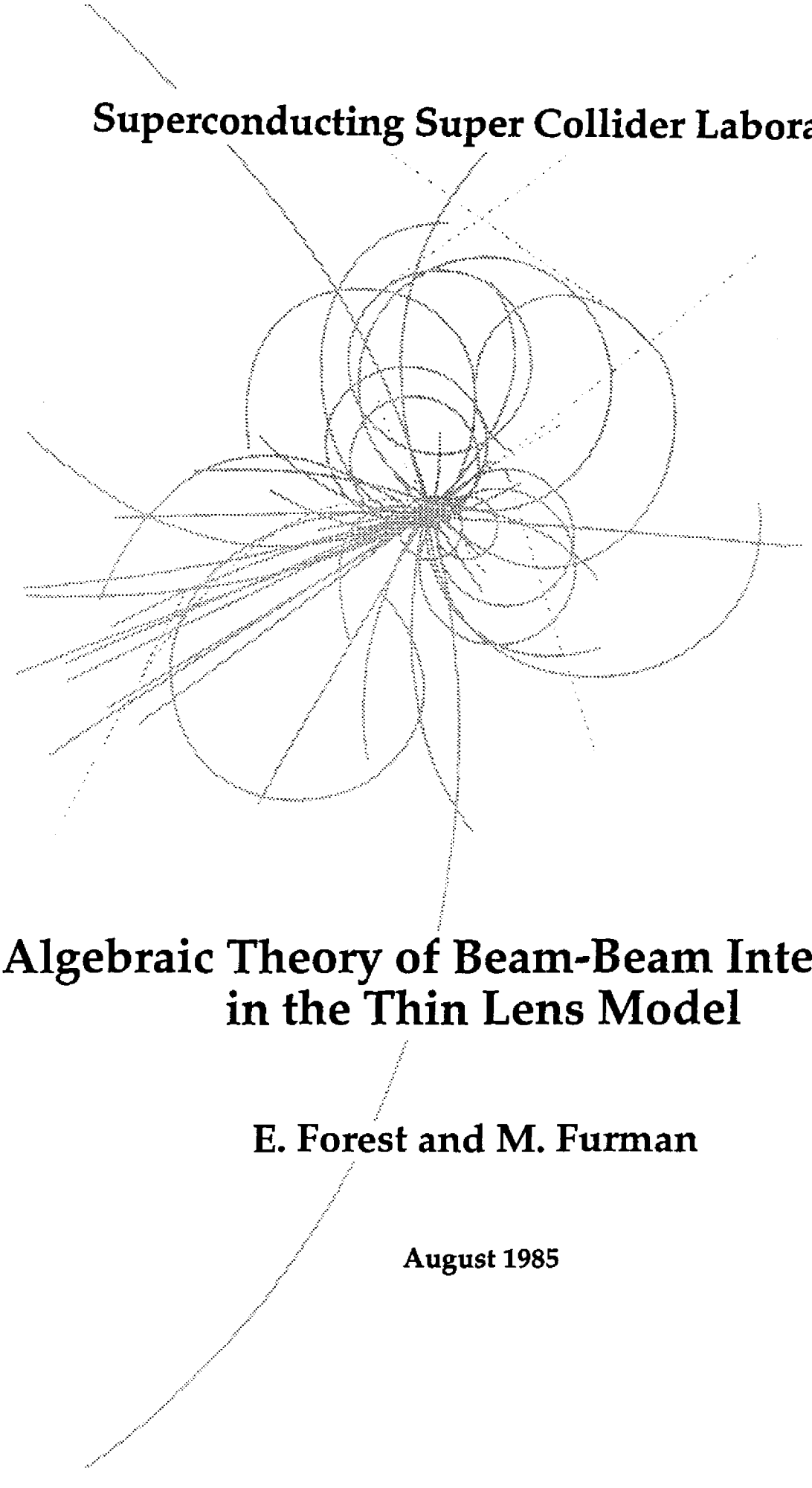

Algebraic Theory of Beam-Beam Interaction in the Thin Lens Model

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ALGEBRAIC THEORY OF BEAM-BEAM INTERACTION IN THE THIN LENS MODEL

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August 1985

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Part I: General Structure of the Problem

1. Introduction

In this paper we will set up a general model for the study of beam-beam interactions.

First, we described the ring as made of N generalized interaction points (or GIP). These GIP's could be single IP or clustered IP's. They are equally spaced.

Then we assign to the i^{th} GIP two generalized bunches (or GB) $\{\vec{\phi}^i, \vec{\psi}^i\}$. These two GB are colliding at the center of the i^{th} GIP. The GB can contain many bunches. This allows for the study of long range effect as well as clustered IP's within the GIP.

Finally, we adopt the Eulerian point of view of hydrodynamic. The label "i" in $\vec{\phi}^i$ or $\vec{\psi}^i$ denotes the GB's at the i^{th} GIP. If $\vec{\phi}^i$ moves toward the right and $\vec{\psi}^i$ toward the left, we see that one collision later $\vec{\phi}^i$ is made of bunches previously at the $(i-1)^{th}$ GIP while $\vec{\psi}^i$ originates from the $(i+1)^{th}$ GIP. (See Fig. 1).

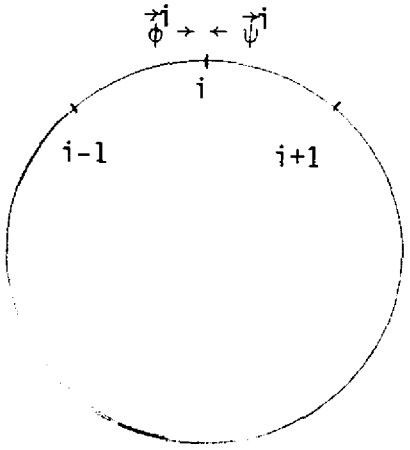


Fig. 1. The ring.

Previously, the Langragian view point was used. Hence one had to follow the GB one full turn. Our scheme simplifies greatly the mathematics.

Now, we are ready to proceed. But before we must comment about $\vec{\phi}^i$ and $\vec{\psi}^i$. These vectors are 2m - vectors representing the dipole mode of the beam:

$$\vec{\phi}^i = (x_0^i, x_0^i, \ldots x_j^i, x_j^{i'}, \ldots, x_{m-1}^i, x_{m-1}^{i'}) \tag{1.1a}$$

$$\vec{\psi}^i = (y_0^i, y_0^{i'}, \ldots, y_j^i, y_j^{i'}, \ldots, y_{m-1}^i, y_{m-1}^{i'}). \tag{1.1b}$$

Furthermore, the are made of canonicallly conjugate pairs:

$$[x_j^i, x_j^{i'}] = 1 \tag{1.2a}$$

$$[y_j^i, y_j^{i'}] = 1 \tag{1.1b}$$

All other Poisson brackets are zero.

In the next section we will make a global statement about the collision operator at a given (GIP).

2. The Global Structure of the Collision Operator H.

Suppose a GB $\vec{\phi}$ collides with $\vec{\psi}$. The collision can be described as a "thin lens" operator acting on $\vec{\phi}$ and $\vec{\psi}$. Let us assume that it is represented by a Lie operator $H(\vec{\phi}, \vec{\psi})$.

$$\bar{\vec{\phi}} = \exp(:H(\vec{\phi},\vec{\psi}):)\vec{\phi} \tag{2.1a}$$

$$\bar{\vec{\psi}} = \exp(:H(\vec{\phi},\vec{\psi}):)\vec{\psi} \tag{2.1b}$$

where $:H:g = [H,g]$.

Consider the canonical transformation which exchanges $\vec{\phi}$ and $\vec{\psi}$:

$$\vec{\psi} = \pi\vec{\phi} \tag{2.2a}$$

$$\vec{\phi} = \pi\vec{\psi} \tag{2.2b}$$

We will impose that π and the collision operator H commute.

$$\pi \exp(:H:)\ \pi = \exp(:H:)$$

$$<=> \exp(:\pi H:) = \exp(:H:)$$

$$<=> \pi H(\vec{\phi}, \vec{\psi}) = H(\vec{\phi}, \vec{\psi})$$

$$<=> H(\vec{\psi}, \vec{\phi}) = H(\vec{\phi}, \vec{\psi}) \qquad (2.3)$$

This is just a statement about the identical character of the two GB $\vec{\phi}$ and $\vec{\psi}$. If we restrict ourself to linear operators (or quadratic H), we can use (2.3) to derive the global structure of the matrix describing the collision. Suppose we have:

$$\bar{\vec{\phi}} = A\vec{\phi} + B\vec{\psi} \qquad (2.4a)$$

$$\bar{\vec{\psi}} = A'\vec{\phi} + B'\vec{\psi} \qquad (2.4b)$$

A, B, A' and B' are 2m-by-2m matrices.

We then apply π on (2.4a):

$$\pi\bar{\vec{\phi}} = (\pi\exp(:H:)\pi)\ \pi\vec{\phi} = \pi A\vec{\phi} + \pi B\vec{\psi} \qquad (2.5)$$

$$<=> \exp(:H:)\bar{\vec{\psi}} = A(\pi\vec{\phi}) + B(\pi\vec{\psi})$$

$$<=> \bar{\vec{\psi}} = A\vec{\psi} + B\vec{\phi}$$

From (2.5) we deduce that A=B' and B=A'. From this result, we can rewrite (2.4) as follows:

$$\begin{pmatrix} \bar{\vec{\phi}} \\ \bar{\vec{\psi}} \end{pmatrix} = A\, e \begin{pmatrix} \vec{\phi} \\ \vec{\psi} \end{pmatrix} + B\delta \begin{pmatrix} \vec{\phi} \\ \vec{\psi} \end{pmatrix} \qquad (2.6a)$$

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (2.6b)$$

The matrices A and B operate the vectors $\vec{\phi}$ and $\vec{\psi}$. They are 2m-by-2m matrices. However e and δ operate identically on all components of $\vec{\phi}$ or $\vec{\psi}$,

they are 2-by-2 matrices. We will see that A and B contain the local information while e and δ are relevant to the global structure or macro-structure.

In the next section, we look at the full ring and we derive the global operator for this collision.

3. The Ring Operator or the Macro-Structure. Consider a ring with N (GIP) labeled from 0 to N-1.

The state of a ring before a collision can be described by the vector $\vec{\Gamma}$:

$$\vec{\Gamma} = (\vec{\phi}^0, \vec{\psi}^0, \ldots \vec{\phi}^{N-1}, \vec{\psi}^{N-1}). \tag{3.1}$$

If a collision takes places, it is described locally in every $(\vec{\phi}^i, \vec{\psi}^i)$ GIP by (2.6). Therefore, we can write its effect on $\vec{\Gamma}$ as follows:

$$\bar{\vec{\Gamma}} = AE\vec{\Gamma} + B\Delta\vec{\Gamma} \tag{3.2a}$$

$$E = \begin{pmatrix} e & & & \\ & e & & \\ & & \ddots & \\ & & & e \end{pmatrix} \quad = 2N\text{-by-}2N \text{ identity} \tag{3.2b}$$

$$\Delta = \begin{pmatrix} \delta & & & \\ & \delta & & \\ & & \ddots & \\ & & & \delta \end{pmatrix} \tag{3.2c}$$

The matrices E and Δ are composed of N small matrices e and δ. Again they commutes with A and B because they act in different spaces. After the collision, we introduce the phase advance R between the GIP:

$$\tilde{\vec{\Gamma}} = R\,\bar{\vec{\Gamma}} = (V\,E + H\,\Delta)\,\vec{\Gamma} \tag{3.3a}$$

$$V = R\,A \tag{3.3b}$$

$$H = R\,B \tag{3.3c}$$

Finally, we must move the various GB's to their new GIP. This is done by the canonical transformation ω:

$$\vec{\phi}^{i\,new} = \omega\tilde{\vec{\phi}}^{i} = \tilde{\vec{\phi}}^{i-1} \tag{3.4a}$$

$$\vec{\psi}^{i\,new} = \omega\tilde{\vec{\psi}}^{i} = \tilde{\vec{\psi}}^{i+1} \tag{3.4b}$$

It is the operator ω which contains the Eulerian viewpoint. If we denote by Ω the 2N-by-2N matrix representation of ω, we can write the matrix T for one full collision as:

$$\vec{\Gamma}^{new} = T\,\vec{\Gamma} \tag{3.5a}$$

$$T = V\Omega + H\Omega\,\Delta \tag{3.5b}$$

If we had followed the G.B. for one turn as it is usually done, we would have obtained the map T^N. Needless to say that T^N is a messy object.

What can we conclude from (3.5)? First we see that powers of T will be made of linear combinations of the elements of the group G generated by $\{\Omega, \Delta\}$. Hence a study of the irreducible representations of G will simplify the study of T. Our goal is to find the stability domain of T as collision parameters are varied. If the study of the irreducible representations reduces the dimensionality of T, then we made great progress. At this stage we will announce the result. If μ labels the μth representation, we have:

$$T\vec{\Gamma}_\mu = (V\Omega_\mu + H\Omega_\mu\Delta_\mu)\,\vec{\Gamma}_\mu, \tag{3.6}$$

where $\vec{\Gamma}_\mu$ has the dimensionality of $(\vec{\phi}, \vec{\psi})$ at most, that is 4m. In fact the group G is isomorphic to a group composed of rotations of angle $2\pi/N$ in the plane and the symmetry along the x-y axis represented by δ.

In the next section, we look at the operator V and H. We will see if we can impose some restrictions on V and H which will further reduce the dimensionality of T.

4. The micro-structure of $H(\vec{\phi},\vec{\psi})$

We are ready to put a tiny bit of physics in the model. Suppose the label "i" refers to individual bunches in the GB. We will assume that they follow each other, i coming before i+1 as shown in Figure 2.

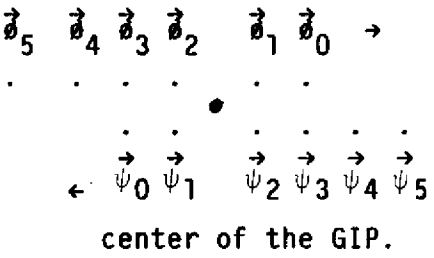


Figure 2 Schematic representation of a collision.

Furthermore we define, without any loss of generality, the vector $\vec{\phi}$ and $\vec{\psi}$ at the center of the GIP. With this convention, the operator R acts identically on each $\vec{\phi}_i$ and $\vec{\psi}_i$. Its matrix representation consists of identical blocks:

$$R = \begin{pmatrix} r & & \\ & r & \\ & & \ddots \\ & & & r \end{pmatrix} \qquad r = \text{2-by-2 matrix} \qquad (4.1)$$

Finally, we made an incorrect statement which we will justify in part III on particular examples. We will assume that H is invariant under the operator defined as follows:

$$t\vec{\phi}_i = \vec{\phi}_{i+1}$$
$$t\vec{\psi}_i = \vec{\psi}_{i+1} \qquad (4.2)$$

In effect we say that the number "m" of bunches in the GB is so large, that the particles in the middle do not see the beginning nor the end. We also impose a cyclic condition which glues the first and the final bunch. This is similar to the Von Karman conditions in a quantized crystal.

Assuming the t-invariance of H, we can use the eigenvectors of t:

$$t\vec{\phi}_\lambda = \lambda\,\vec{\phi}_\lambda$$
$$t\vec{\psi}_\lambda = \lambda\vec{\psi}_\lambda \qquad (4.3)$$

In this basis the collision operator is block diagonal. Each block is made of the $(\vec{\phi}_\lambda, \vec{\psi}_\lambda)$ pair. Equation (2.6) can be rewritten as follows:

$$\begin{pmatrix} \vec{\phi} \\ \hline \vec{\psi} \end{pmatrix} = \exp(:H:)\begin{pmatrix} \vec{\phi}_\lambda \\ \vec{\psi}_\lambda \end{pmatrix} = a_\lambda \begin{pmatrix} \vec{\phi}_\lambda \\ \vec{\psi}_\lambda \end{pmatrix} + b_\lambda \delta \begin{pmatrix} \vec{\phi}_\lambda \\ \vec{\psi}_\lambda \end{pmatrix} \qquad (4.4)$$

Basically, we have written the matrices A and B in term of the irreducible representations of the group $C_m = \{t^\alpha \mid \alpha = 0, m-1\}$. The group C_m is isomorphic to the group formed by 2-dimensional rotation of angle $2\pi/m$.

5. The micro and macro-structure of the problem.

The group structure of the full ring is now simple. It consists of the ring group $G = \{\Omega, \Delta\}$ and the local GB group C_m. The full ring group is then $G \otimes C_m$.

Combining the results of section III and IV, we can rewrite the map T with the help of (3.6) and (4.4).

$$T\,\vec{\Gamma}_{\mu,\lambda} = (r\,a_\lambda\,\Omega_\mu + rb_\lambda\Omega_\mu\,\Delta_\mu)\,\vec{\Gamma}_{\mu,\lambda} \qquad (5.1a)$$

$$\vec{\Gamma}_{\mu,\lambda} = (\vec{\phi}_{\mu,\lambda}, \vec{\psi}_{\mu,\lambda}),\ V_\lambda = ra_\lambda,\ H_\lambda = rb_\lambda \qquad (5.2b)$$

The vector $\vec{\Gamma}_{\mu,\lambda}$, is at most a four vector. The dimensionality of the problem has been greatly reduced.

In the next part of this paper, we examine the irreducible representations of G and C_m.

Part II: The Ring Group $G \otimes C_m$.

I. Introduction

Before looking at actual models, we need to understand the structure of the operators involved. In part I, we have define two groups relevant to the problem. The most trivial one is the abelian group C_m. Its multiplication table is given by:

$$t^{\alpha}t^{\beta} = t^{\alpha+\beta} \qquad \alpha+\beta \bmod [m]. \tag{1.1}$$

This group has m irreducible one-dimensional representations labeled by λ. In fact the homomorphism f_λ connecting t^α to its representation is just:

$$f_\lambda(t^\alpha) = \exp\left(\frac{i2\pi}{m}\lambda\alpha\right) \tag{1.2}$$

One sees the triviality of the group C_m.

On the other hand G is non-abelian and therefore more complex. The next sections will be devoted almost entirely to G and the macro-structure of the operator T.

2. The Group Table of G

We first start with the map ω as given by (I.3.4). What is ω^α?

$$\omega^\alpha \vec{\phi}^i = (\omega^{\alpha-1})\ \omega\vec{\phi}^i = \omega^{\alpha-1}\vec{\phi}^{\,i-1} = \vec{\phi}^{\,i-\alpha} \tag{2.1a}$$

$$\omega^\alpha \vec{\psi}^i = (\omega^{\alpha-1})\ \omega\ \vec{\psi}^i = \omega^{\alpha-1}\ \vec{\psi}^{i+1} = \vec{\psi}^{i+\alpha} \tag{2.1b}$$

Clearly, the operation $\omega^{\alpha}\ \omega^{\beta}$ is just $\omega^{\alpha+\beta}$. Furthermore, since the index "i" is defined modulo N, we must have $\omega^{N} = E$ where E is the group identity. As expected ω alone generates C_N. This is not surprising since the purpose of ω as defined by (I.3.4) is to rotate the beam to the next G.I.P. Since a full turn corresponds to "N" G.I.P., we must come back to the original G.I.P.

As we know G is more complex then C_N because of operator Δ.

The effect of Δ is to exchange $\vec{\phi}^{i}$ and $\vec{\psi}^{i}$:

$$\Delta\ \vec{\psi}^{i} = \vec{\phi}^{i} \tag{2.2a}$$

$$\Delta\ \vec{\phi}^{i} = \vec{\psi}^{i} \tag{2.2b}$$

We notice that Δ is involutive:

$$\Delta^{2} = E \tag{2.3}$$

We will show that the other elements in the group G are all amenable to the form $\Delta\omega^{\alpha}$. We notice that $\omega^{-\alpha}\Delta$ is the inverse of $\Delta\omega^{\alpha}$:

$$\Delta\omega^{\alpha}\ \omega^{-\alpha}\Delta = \Delta^{2} = E \tag{2.4}$$

Now, we will show that $\Delta\omega^{\alpha}$ is $\omega^{-\alpha}\Delta$. In other words $\Delta\omega^{\alpha}$ is also an involution. Using the definition of Δ and ω we have:

$$\begin{array}{ccccc} & & \overset{\Delta\omega^{\alpha}}{\longrightarrow} & & \\ & \overset{\omega^{\alpha}}{\longrightarrow} & & \overset{\Delta}{\longrightarrow} & \\ \vec{\phi}^{i} & \rightarrow & \vec{\phi}^{i-\alpha} & \rightarrow & \vec{\psi}^{i-\alpha} \\ \vec{\psi}^{i} & \rightarrow & \vec{\psi}^{i+\alpha} & \rightarrow & \vec{\phi}^{i+\alpha} \end{array} \tag{2.5}$$

If we apply $\Delta\omega^{\alpha}$ twice, we deduce from (2.5):

$$\begin{array}{ccccc} & \overset{\Delta\omega^{\alpha}}{\longrightarrow} & & \overset{\Delta\omega^{\alpha}}{\longrightarrow} & \\ \vec{\phi}^{i} & \rightarrow & \vec{\psi}^{i-\alpha} & \rightarrow & \vec{\phi}^{i} \\ \vec{\psi}^{i} & \rightarrow & \vec{\phi}^{i+\alpha} & \rightarrow & \vec{\psi}^{i} \end{array} \quad \Rightarrow \quad (\Delta\omega^{\alpha})^{2} = E \tag{2.6}$$

Combining (2.6) and (2.4), we see how one can simplify a product of Δ's and ω's:

$$\omega^{\beta}\Delta\omega^{\alpha} = \Delta\omega^{\alpha-\beta} \tag{2.7}$$

Finally we can display the full group:

$$G = \left\{ x \;\middle|\; x = \omega^{\alpha},\ x = \Delta\omega^{\alpha};\ \alpha = 0,\ N-1 \right\} \tag{2.8}$$

The order of the group is 2N. In the next section, we derive the classes of equivalence of G.

3. The Classes of G

Consider the following equivalence relation of the group:

$$x\ P\ y \iff \exists a \in G;\ a\ x\ a^{-1} = y\ . \tag{3.1}$$

First we look at ω^{β}:

$$\begin{aligned} \Delta\omega^{\alpha}\ \omega^{\beta}\ \omega^{-\alpha}\Delta &= \Delta\omega^{\beta}\Delta \\ &= \Delta^{2}\ \omega^{-\beta} \\ &= \omega^{-\beta} \\ \Rightarrow\ \omega^{\beta} P\ \omega^{-\beta}\ . & \end{aligned} \tag{3.2}$$

Secondly, we look at the class of $\Delta\omega^{\beta}$.

$$\omega^{\alpha}\Delta\omega^{\beta}\omega^{-\alpha} = \Delta\omega^{\beta-2\alpha} \tag{3.3a}$$

$$\Delta\omega^{\alpha}\Delta\omega^{\beta}\Delta\omega^{\alpha} = \Delta\omega^{-\beta+2\alpha} \tag{3.3b}$$

$$\Rightarrow\ \Delta\omega^{\beta}\ P\ \Delta\omega^{\beta-2\alpha}\ P\ \Delta\omega^{-\beta+2\alpha}\ .$$

Using the results of (3.2) and (3.3), we distinguish two cases, N odd and N even.

N odd: (N+3)/2 classes.

$$G = \{E\} \cup \{\Delta\omega^{\alpha};\ \alpha = 0,\ N-1\} \bigcup_{\beta=1,\ \frac{N-1}{2}} \{\omega^{\beta},\ \omega^{-\beta}\}, \qquad (3.4)$$

N even: (N+6)/2 classes.

$$G = \{E\} \cup \{\Delta\omega^{2\alpha};\ \alpha = 0,\ \tfrac{N}{2} - 1\} \cup \{\Delta\omega^{2\alpha+1};\ \alpha = 0,\ \tfrac{N}{2} - 1\} \bigcup_{\beta=1,\ \frac{N-2}{2}} \{\omega^{\beta},\ \omega^{-\beta}\}$$

$$\cup \{\omega^{N/2}\}. \qquad (3.5)$$

The number of classes is equal to the number of irreducible representations. In the next section, we will give the irreducible representations.

4. The Irreducible Representations of G

In the next tables, we list the representations for the two cases.

N odd	E	ω	Δ	Multiplicity	Total Dimensionality
0-Mode	1	1	1	1	1
π-Mode	1	1	-1	1	1
μ-Mode	E_μ=e	Ω_μ	$\Delta_\mu=\delta$	2	$2\times 2 \times \frac{(N-1)}{2}$

$$\Omega_\mu = \begin{pmatrix} \cos(\frac{2\pi}{N}\mu) & \sin(\frac{2\pi}{N}\mu) \\ -\sin(\frac{2\pi}{N}\mu) & \cos(\frac{2\pi}{N}\mu) \end{pmatrix} \quad e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mu = 1,\ \frac{N-1}{2}$$

N Even	E	ω	Δ	Multiplicity	Total Dimensionality
0-Mode	1	1	1	1	1
π-Mode	1	1	-1	1	1
0^--Mode	1	-1	1	1	1
π^--Mode	1	-1	-1	1	1
μ-Mode	$E_\mu = e$	Ω_μ	$\Delta_\mu = \delta$	2	$2 \times 2 \times \frac{(N-2)}{2}$

$\mu = 1, \frac{N-2}{2}$

The table shows the multiplicity of every representation in the original 2N - representation. Indeed the total dimensionality adds up to 2N. As advertized in section (I.3), the matrices Ω_μ and Δ_μ are two dimensional at most.

For completeness, we list some famous relations of group theory. With these, the reader can check the table of irreducible representations. If $X^\nu(x)$ is the character (Trace) of x in the ν^{th} irreducible representation, then the following formulae are true:

$$\sum_{x \epsilon G} X^\nu(x) X^{\mu *}(x) = g \delta_{\nu\mu} \tag{4.1}$$

g = order of G = 2N.

$$a_\mu = \frac{1}{g} \sum_{x \epsilon G} X(x)\, X^\mu(x)^* \tag{4.2}$$

X(x) = Trace of x in the original 2N by 2N representation.

Finally, we notice that our original representation is the so-called regular representation. It obeys the following relations:

$$X(x) = \begin{cases} 0 \text{ for } x \neq E \\ g = 2N \text{ for } x = E. \end{cases} \tag{4.3}$$

With (4.1), (4.2) and (4.3), the reader can now check the tables. In the next section, we rederive these group theoretical results with the help of discrete Fourier transform.

5. The Transform Method

With this method, we must write the map T in component form. Using (I.2.6) and (I.3.3) we get:

$$\vec{\Gamma} = (V\,E + H\Delta)\,\vec{\Gamma} \qquad \vec{\Gamma} = (\vec{\phi}^0, \vec{\psi}^0 \ldots\ldots, \vec{\phi}^{N-1}, \vec{\psi}^{N-1}) \tag{5.1a}$$

$$\vec{\phi}^i = V\,\vec{\phi}^i + H\,\vec{\psi}^i \tag{5.1b}$$

$$\vec{\psi}^i = H\,\vec{\phi}^i + V\,\vec{\psi}^i \tag{5.1c}$$

and, finally, with the help of (I.3.4) and (I.3.5):

$$\vec{\phi}^{i+1 \text{ new}} = \vec{\phi}^i = V\,\vec{\phi}^i + H\,\vec{\psi}^i \tag{5.2a}$$

$$\vec{\psi}^{i-1 \text{ new}} = \vec{\psi}^i = H\,\vec{\phi}^i + V\,\vec{\psi}^i \tag{5.2b}$$

Equation (5.2) is the equivalent of (I.3.5). To solve it we define a cosine-sine transform as follows:

$$\underset{\sim}{\phi}^{\mu} = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \left\{ \cos\left(\frac{2\pi\mu i}{N}\right) + \sin\left(\frac{2\pi\mu i}{N}\right) \right\} \vec{\phi}^i \,. \tag{5.3}$$

Notice that this transformation is invertible:

$$\vec{\phi}^i = \sum_{\mu=0}^{N-1} \varphi\left(\frac{2\pi\mu i}{N}\right) \underset{\sim}{\phi}^{\mu}, \quad \varphi(x) = (\cos x + \sin x)/\sqrt{N}. \tag{5.4a}$$

We can derive one useful relation:

$$\sum_{i=0}^{N-1} \vec{\phi}^{i+\alpha}\, \varphi\left(\frac{2\pi\mu i}{N}\right) = \sum_{i=0}^{N-1} \vec{\phi}^i\, \varphi\left(\frac{2\pi\mu(i-\alpha)}{N}\right)$$

$$= \cos\left(\frac{2\pi\mu\alpha}{N}\right) \underset{\sim}{\phi}^{\mu} - \sin\left(\frac{2\pi\mu\alpha}{N}\right) \underset{\sim}{\phi}^{-\mu} \tag{5.4b}$$

μ is obviously a modulo N number.

The transform Kernel $\varphi(x)$ is defined so that our transformation is real. It will provide an easy connection to the group theory derivation of the previous section.

We proceed by transforming equation (5.2):

$$C \underset{\sim}{\overline{\phi}}^{\mu} - S \underset{\sim}{\overline{\phi}}^{-\mu} = V \underset{\sim}{\phi}^{\mu} + H \underset{\sim}{\psi}^{\mu} \tag{5.5a}$$

$$C \underset{\sim}{\overline{\psi}}^{\mu} + S \underset{\sim}{\overline{\psi}}^{-\mu} = H \underset{\sim}{\phi}^{\mu} + V \underset{\sim}{\psi}^{\mu} \tag{5.5b}$$

We have change the superscript "New" by a bar for convenience. Notice that μ and $-\mu$ are coupled. Therefore we must rewrite (5.a,b) with the opposite sign:

$$C \underset{\sim}{\overline{\phi}}^{-\mu} + S \underset{\sim}{\overline{\phi}}^{\mu} = V \underset{\sim}{\phi}^{-\mu} + H \underset{\sim}{\psi}^{-\mu} \tag{5.5c}$$

$$C \underset{\sim}{\overline{\psi}}^{-\mu} - S \underset{\sim}{\overline{\psi}}^{\mu} = H \underset{\sim}{\phi}^{-\mu} + V \underset{\sim}{\psi}^{-\mu} \tag{5.5d}$$

By suitable adding and substracting, we can decouple (5.5) in two pairs. Consider the vectors $(\sigma_{\pm}, \gamma_{\pm})$ defined to be

$$\underset{\sim}{\sigma}_{\pm}^{\mu} = (\underset{\sim}{\phi}^{\mu} \pm \underset{\sim}{\psi}^{-\mu})/\sqrt{2} \tag{5.6a}$$

$$\underset{\sim}{\gamma}_{\pm}^{\mu} = (\underset{\sim}{\phi}^{-\mu} \pm \underset{\sim}{\psi}^{\mu})/\sqrt{2} \ . \tag{5.6b}$$

In this basis, (5.5) becomes:

$$\Omega_{-\mu} \begin{pmatrix} \underset{\sim}{\overline{\sigma}}_{\pm}^{\mu} \\ \underset{\sim}{\overline{\gamma}}_{\pm}^{\mu} \end{pmatrix} = (V\, e \pm H\delta) \begin{pmatrix} \underset{\sim}{\sigma}_{\pm}^{\mu} \\ \underset{\sim}{\gamma}_{\pm}^{\mu} \end{pmatrix} . \tag{5.7}$$

This equation can be made to look like (I.3.6) by a simple rotation:

$$\vec{V}_{1}^{\mu} = \begin{pmatrix} \sigma_{+}^{\mu} \\ \gamma_{+}^{\mu} \end{pmatrix}$$

$$\vec{V}_{2}^{\mu} = \Omega_{\frac{N}{4}} \begin{pmatrix} \sigma_{-}^{\mu} \\ \gamma_{-}^{\mu} \end{pmatrix} \tag{5.8}$$

Then, equation (5.7) takes the form:

$$\vec{\bar{V}}_{i}^{\mu} = (V\Omega_{\mu} + H\,\Omega_{\mu}\delta)\,\vec{V}_{i}^{\mu} \quad i = 1,2 \; . \tag{5.9}$$

$$0 \leq \mu \leq \frac{N}{2}$$

Connection can be made with the group theoretical treatment. For $\mu \neq 0 \neq N/2$, we have two μ-modes as shown in the table of section 4. For $\mu = 0$, $\sigma_{\pm}^{0}$ and $\gamma_{\pm}^{0}$ are identical. Inserting these in equation (5.7), we regain the one dimensional 0 and π modes. Furthermore, if N is even, we obtain two additional one-dimensional modes, namely 0^{-} and π^{-}. These occur at $\mu = N/2$.

The "Transform" method gave us an explicit form for the modes. The group G provided us with an understanding of the macro-structure of our problem. In Part III of this paper we will look at the micro-structure of some collisions. We will discover why the assumptions of section (I.4) are not too unrealistic. For some special collisions we will derive the operators V_{λ} and H_{λ} of equation (5.1).

Part III. Examples of the beam-beam interactions.

1. Introduction

In the last part of this paper we will display some analytical computations and the corresponding computer simulations. In section 2, we review the assumption used to set up the dipole approximation of the beam-beam problem. In section 3, we derive the operator :H: of section (I.2). In section 4 we derive the stability curve in the absence of long-range interaction. In sections 5 and 6 we introduce two models for the phase advance between the long-range collisions.

Finally, we conclude with a comparison of the analytic prediction and the corresponding simulation.

2. A short review of the physical origin of the model

We consider here the dipole motion of the bunches in one dimension only. Each bunch is described by its transverse coordinate x and slope x'. The bunches are evenly spaced and populate the beams uniformly. Both beams are identical. All the IR's are also identical, as are the arcs between them.

Thus the configuration of the machine is described by the number of IR's, the number of bunches within an IR and the number of bunches within an IR and the number of bunches within an arc. The number of bunches per beam is

$$N_b = N(m_i + m_a) \tag{2.1}$$

where N is the number of IR's, m_i is the maximum number of bunches that can fit simultaneously within an IR and m_a is the number of bunches that can fit simultaneously within an arc.

The bunches interact only within the IR's. A given bunch interacts every time it moves a distance L/2 where L is the interbunch distance. Within an IR bunches drift between collisions. In the arcs, bunches are transported by a phase advance matrix from the end of an IR to the beginning of the next.

There is one "head-on" collision at the center of the IR plus several "long-range" interactions on the sides. We assume these interactions to be kicks. Consider two opposing bunches whose coordinates, measured relative to their respective design trajectories, are (x, x') and (y, y'). They collide at a point where the distance between the design trajectories is d. The slopes change according to $x' \to x' + \Delta x'$ and $y' \to y' - \Delta x'$ where

$$\Delta x' = \frac{1}{f}[E(d+\Delta x, \sigma) - E(d,\sigma)]$$

$$\Delta x = x - y$$

$$E(x,\sigma) = \frac{2\sigma^2}{x}[1-\exp(-\frac{x^2}{2\sigma^2})] \qquad (2.2)$$

(we assume a Gaussian charge distribution of size σ). The parameter f is related to the bunch characteristics by

$$\frac{1}{f} = \frac{Nr_o}{\gamma\sigma^2} \qquad (2.3)$$

where N is the number of particles per bunch, r_o is the classical radius of the particle and γ is the usual relativistic factor.

For the purposes of our study, we linearize the kicks about $\Delta x=0$. This requires the reasonable assumptions $d \gg |\Delta x|$ for the long-range kicks and $|\Delta x| \ll \sigma$ for the head-on kick. A straightforward expression gives

$$\Delta x' = \frac{1}{f^*}\Delta x \qquad \text{(head-on)}$$

$$\Delta x' = + \frac{1}{f^*} \frac{P}{s^2} \Delta x \quad \text{(long-range)}$$

$$P = -2(\sigma^*/\alpha)^2$$

The distance from the interaction point is denoted by "s". The quantity σ^* is the width of the beam at the interaction point.

In our simulation we used the parameter of the design study of the SSC.

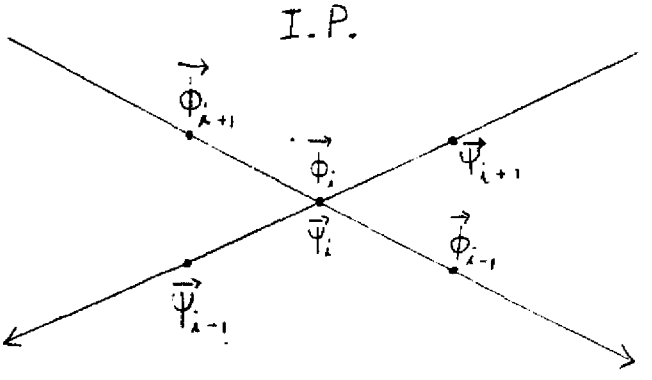
$$P = -0.039 \text{ for } \alpha = 50 \ \mu\text{rad} \qquad (2.5)$$

$$P = -0.984 \text{ for } \alpha = 10 \ \mu\text{rad}.$$

The interbunch spacing was set at 15 meters.

3. The micro-collision operator H

Suppose the ring as N interaction sites and each beam has mN bunches. The picture we have in mind is the one of section (I.4). Let us assume for the moment that there is no long range interaction.



Notice that all the $\vec{\phi}_j$ and $\vec{\psi}_j$ are expressed at the I.P. (Center of what we call a GIP).

In this case the Lie operator for one collision between two G.B. is:

$$M = \exp(:\tfrac{\epsilon}{2}(x_0-y_0)^2:)\ldots\exp(:\tfrac{\epsilon}{2}(x_{m-1}-y_{m-1})^2:) \tag{3.1a}$$

$$M = \exp(:\tfrac{\epsilon}{2}\sum_{j=0}^{m-1}(x_j-y_j)^2:) \tag{3.1b}$$

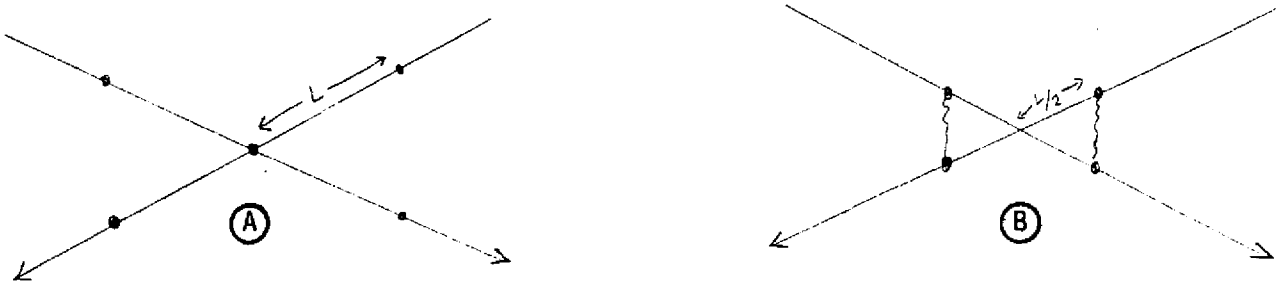
$$H^0 = \tfrac{\epsilon}{2}\sum_{j=0}^{m-1}(x_j-y_j)^2 = \sum_{j=0}^{m-1} H_j \tag{3.1c}$$

In (3.1a), we expressed M as product of successive collisions. However, since the exponents commute we can join them. As a result M is exactly given by (3.1b). Notice that H^0 is translationally invariant:

$$tH^0 = \tfrac{\epsilon}{2}\sum_{j=0}^{m-1}(tx_j-ty_i)^2$$

$$= \tfrac{\epsilon}{2}\sum_{j=0}^{m-1}(x_{j+1}-y_{j+1})^2 = H^0 \tag{3.2}$$

Now, let us put one long range interaction at a distance L/2:



The collision (A) can be described by the operator H_i defined as follows.

$$H_i = \tfrac{\epsilon}{2}(x_i-y_i)^2 \tag{3.3}$$

However, the collision (B) happens between phase advanced quantities. Let us call σ_1 the operator which advances the bunch to the I.P. similarly,

ρ_1, takes the bunch away from the I.P.. In terms of these operators, B is described by the operator H_{jj+1}.

$$H_{jj+1} = \frac{\varepsilon'}{2} \left\{(\rho_1 x_j - \sigma_1^{-1} y_{j+1})^2 + (\rho_1 y_j - \sigma_1^{-1} x_{j+1})^2\right\} \tag{3.4}$$

In terms of H_j and H_{jj+1}, the map M for a collision is given by:

$$M = \exp(:H_0:)\exp(:H_{01}:)\exp(:H_1:)\ldots\exp(:H_{m-1}:). \tag{3.5}$$

Here, we introduce our first approximation. We decide to join all the exponents. Within a G.B., we neglect products of ε and ε^1.

$$M = \exp(: \sum_{j=0}^{m-1} H_j + \sum_{j=0}^{m-1} H_{jj+1} :) \tag{3.6}$$

Furthermore, we added $H_{m-1\ 0}$ to (3.6). This is similar to the Von Karman boundary condition of cristallography where end effects are neglected. Finally we can write:

$$H^1 = \sum_{j=0}^{m-1} H_j + \sum_{j=0}^{m-1} H_{jj+\ell} + \ldots \text{Order 2}. \tag{3.7}$$

Clearly, if we include J long range collisions, equation (3.7) can be generalized:

$$H^J = \sum_{j=0}^{m-1} H_j + \sum_{\ell=1}^{J} \sum_{j=0}^{m-1} H_{jj+\ell} \tag{3.8a}$$

$$H_{jj+\ell} = \frac{\varepsilon}{2}^{[\ell]} \left\{(\rho_\ell x_j - \sigma_\ell^{-1} y_{j+\ell})^2 + (\rho_\ell y_j - \sigma_\ell^{-1} x_{j+\ell})^2\right\} . \tag{3.8b}$$

By construction our collision operator H is invariant under the group C_m. Hence, even if σ_ℓ and ρ_ℓ are complicated operators the T operator will fully decompose following equation (I.5.1).

In the next section, we study the simplest operator, H^0. In this case where we neglect all long range interactions.

4. The stability curves for an H^0 ring.

Without any long range interactions, the collision is controlled by H^0. In fact, M as given by (3.1b) is a kick:

$$\bar{x}_j = x_j$$

$$\bar{x}'_j = x'_j + [H^0, x'_j] = x'_j + \varepsilon(x_j - y_j) \tag{4.1}$$

Notice that the operation is block diagonal in the variables (x_j, x'_j, y_j, y'_j). Following (I.4.4), we can write immediately a_j and b_j.

$$a_j = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix} \qquad b_j = \begin{pmatrix} 0 & 0 \\ -\varepsilon & 0 \end{pmatrix} . \tag{4.2}$$

Notice that a_j and b_j are not functions of j. The problem is highly degenerate as we might have expected. This result is substituted in equation (I.5.1):

$$T_\mu = ra\ \Omega_\mu + rb\ \Omega_\mu \Delta \tag{4.3a}$$

$$r = \begin{pmatrix} \cos(2\pi\upsilon) & \sin(2\pi\upsilon) \\ -\sin(2\pi\upsilon) & \cos(2\pi\upsilon) \end{pmatrix} = \begin{pmatrix} c\upsilon & s\upsilon \\ -s\upsilon & c\upsilon \end{pmatrix} \tag{4.3b}$$

The matrix T_μ is a 4-by-4 symplectic matrix. Its characteristic polynomial will give us the tune of the matrix T_μ. Denoting by $P(\lambda)$ the polynomial $\mathrm{Det}(T-\lambda I)$, we compute the following quantities:

$$\alpha_\mu = P(1) + P(-1)$$

$$\beta_\mu = P(1) - P(-1) \tag{4.4}$$

The tune Θ_μ of T_μ can computed from (4.4) using the formula:

$$\cos\Theta_\mu = \frac{-\beta_\mu}{16} \pm \sqrt{\left(\frac{\beta_\mu}{16}\right)^2 - \frac{\alpha_\mu}{8} + 1}\ . \tag{4.5}$$

The stability limit is reached at $\cos\theta_\mu = \pm 1$. Substitution of this limit in (4.5) leads to:

$$\pm \beta_\mu + \alpha_\mu = 0 \ . \tag{4.6}$$

In the case of H^0, we compute α_μ^0 and β_μ^0:

$$\alpha_\mu^0 = 8(c_\mu{}^2 + c_\nu{}^2 + \epsilon\, c_\nu s_\nu) \tag{4.7a}$$

$$\beta_\mu^0 = -8(2c_\mu c_\nu + \epsilon\, c_\mu s_\nu) \tag{4.7b}$$

$$c_\mu = \cos\left(\frac{2\pi\mu}{N}\right) \qquad s_\mu = \sin\left(\frac{2\pi\mu}{N}\right) \tag{4.7c}$$

Solving for $\xi = \epsilon/4\pi$, one gets:

$$\xi_\pm = -\frac{1}{4\pi}\ \frac{(c_\nu \mp c_\mu)}{s_\nu} \ . \tag{4.8}$$

If the parameter $\xi \pm$ takes values greater than what is permitted by (4.8), the T_μ matrix is unstable. The superposition of $\xi(\nu)$ for all the values of μ will determine the stability curve.

We found perfect agreement between the simulation and the result of (4.8). This is expected since <u>no</u> approximations were made in computing the collision map H^0.

In the next section, we look at a class of H^J's which is solvable in terms of a simple quadratic in ξ.

5. A simple model for H^J, the telescopic approximation.

As we have seen the strength parameter $\varepsilon^{[\ell]}$ goes down as the second power of the distance from the I.P. In fact, we can write $\varepsilon^{[\ell]}$ in terms of the main collision parameter ε:

$$\varepsilon^{[\ell]} = \frac{P}{\ell^2 (L/2)^2} \varepsilon \tag{5.1}$$

L = interbunch spacing

The correct expression for $H_{jj+\ell}$ is given by (3.8b). We can simulate the effect of the transport maps ρ_ℓ and σ_ℓ by a modification of the parameter $\varepsilon^{[\ell]}$.

$$H_{jj+\ell} = \frac{\varepsilon^{[\ell]}}{2} (\rho_\ell x_j - \sigma_\ell^{-1} y_{j+\ell})^2 + (\rho_\ell y_j - \sigma_\ell^{-1} x_{j+\ell})^2$$

$$\underline{\text{is replaced by}} \quad \frac{\varepsilon^{[\ell]}}{2} \beta(\tfrac{\ell L}{2}) \left\{ (x_j - y_{j+\ell})^2 + (y_j - x_{j+\ell})^2 \right\} \tag{5.2}$$

Here we assume that the collisions at $s = \ell L/2$ are of the same nature as the main collisions. The only effect of the operators ρ_ℓ and σ_ℓ^{-1} is to magnify the rays.

In fact, equation (5.2) is exact if ρ_ℓ and σ_ℓ^{-1} are given by the matrix representation:

$$\rho_\ell = \sigma_\ell^{-1} = \begin{pmatrix} \sqrt{\beta} & 0 \\ 0 & 1/\sqrt{\beta} \end{pmatrix}. \tag{5.3}$$

In a real case, ρ_ℓ and σ_ℓ^{-1} are closer to drifts than the telescopic map (5.3).

Let us proceed with the computation of the matrices a_λ and b_λ. Notice that H^J will be a kick operator:

$$\bar{x}_j = x_j \tag{5.4a}$$

$$\bar{x}'_j = x'_j + [H^J, x'_j] \tag{5.4b}$$

$$= x'_j + \epsilon(x_j - y_j) + \sum_{\ell=1}^{J} \epsilon^{[\ell]}\beta_\ell \left\{ 2x_j - y_{j+\ell} - y_{j-\ell} \right\}.$$

Equation (5.4b) can be partially decoupled by a canonical complex Fourier transform:

$$\underset{\sim}{x}_\lambda = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} \exp(i\tfrac{2\pi}{m}\lambda j)\, x_j \tag{5.5a}$$

$$\underset{\sim}{x}'_\lambda = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} \exp(-i\tfrac{2\pi}{m}\lambda j)\, x'_j \tag{5.5b}$$

$$[\underset{\sim}{x}_\lambda, \underset{\sim}{x}'_\lambda] = 1 \quad .$$

Using (5.5), we can rewrite (5.4):

$$\bar{\underset{\sim}{x}}'_\lambda = \underset{\sim}{x}'_{-\lambda} + (\epsilon+d\epsilon)\underset{\sim}{x}_\lambda - (\epsilon+\epsilon q(\lambda))\underset{\sim}{y}_\lambda, \tag{5.6a}$$

$$q(\lambda) = \frac{2P}{(L/2)^2} \sum_{\ell=1}^{J} \cos(\tfrac{2\pi}{m}\lambda\ell)\, \frac{\beta_\ell}{\ell^2}, \tag{5.6a}$$

$$d = q(0)$$

Notice that λ and $-\lambda$ are coupled. This can be removed by a canonical transformation of the type presented in (II.5.6):

$$\underset{\sim}{x}^{\pm}_{\lambda} = (\underset{\sim}{x}_{\lambda} \pm \underset{\sim}{x}_{-\lambda})/\sqrt{2} \tag{5.7a}$$

$$x^{\pm\prime}_{\lambda} = (\underset{\sim}{x}'_{\lambda} \pm \underset{\sim}{x}'_{-\lambda})/\sqrt{2} \tag{5.7b}$$

In terms of $x^{\pm}_{\lambda}$ and $y^{\pm}_{\lambda}$, equation (5.6) takes its final form:

$$\overline{\underset{\sim}{x}}^{\pm}_{\lambda} = \underset{\sim}{x}^{\pm}_{\lambda} \qquad \overline{\underset{\sim}{y}}^{\pm}_{\lambda} = \underset{\sim}{y}^{\pm}_{\lambda} \ , \tag{5.8a}$$

$$\overline{\underset{\sim}{x}}^{\pm\prime}_{\lambda} = \underset{\sim}{x}^{\pm\prime}_{\lambda} + \epsilon(1+d)\ \underset{\sim}{x}^{\pm}_{\lambda} - \epsilon(1+q(\lambda))\ \underset{\sim}{y}^{\pm}_{\lambda} \tag{5.8b}$$

$$\overline{\underset{\sim}{y}}^{\pm\prime}_{\lambda} = \underset{\sim}{y}^{\pm\prime}_{\lambda} + \epsilon(1+d)\ \underset{\sim}{y}^{\pm}_{\lambda} - \epsilon(1+q(\lambda))\ \underset{\sim}{x}^{\pm}_{\lambda} \ .$$

From (5.8), we can extract a_{λ} and b_{λ}:

$$a_{\lambda} = \begin{pmatrix} 1 & 0 \\ \epsilon(1+d) & 1 \end{pmatrix} \qquad b_{\lambda} = \begin{pmatrix} 0 & 0 \\ -\epsilon(1+q(\lambda)) & 0 \end{pmatrix}. \tag{5.9}$$

The quantities α^{λ}_{μ} and β^{λ}_{μ} of (4.6) can be computed exactly:

$$\alpha^{\lambda}_{\mu} = \alpha^{0}_{\mu} + s_{\upsilon}^{2}\ (-4q-2q^{2}+2d^{2}+4d)\ \epsilon^{2}+8c_{\upsilon}s_{\upsilon}d\epsilon \tag{5.10a}$$

$$\beta^{\lambda}_{\mu} = \beta^{0}_{\mu} - 8\ c_{\mu}s_{\upsilon}d\epsilon \ , \tag{5.10b}$$

where α^{0}_{μ} and β^{0}_{μ} are the H^{0} results of equation (4.7).

Finally, we must solve the stability equation:

$$\pm\beta^{\lambda}_{\mu} + \alpha^{\lambda}_{\mu} = 0 \Leftrightarrow 8(c_{\upsilon}+\bar{c}_{\mu})^{2} + 8(1+d)s_{\upsilon}(c_{\upsilon}+\bar{c}_{\mu})\epsilon+2\left\{d^{2}-q^{2}+2(d-q)\right\}s_{\upsilon}^{2}\epsilon^{2} = 0 \tag{5.11}$$

Remarkably, the solutions of (5.11) are expressible in terms of $\xi_{\pm}$ of line (4.8):

$$\xi^1_{\pm} = \left(\frac{2}{2+d+q(\lambda)}\right) \quad \xi_{\pm} \quad , \tag{5.12a}$$

$$\xi^2_{\pm} = \left(\frac{2}{d-q(\lambda)}\right) \quad \xi_{\pm} \quad . \tag{5.12b}$$

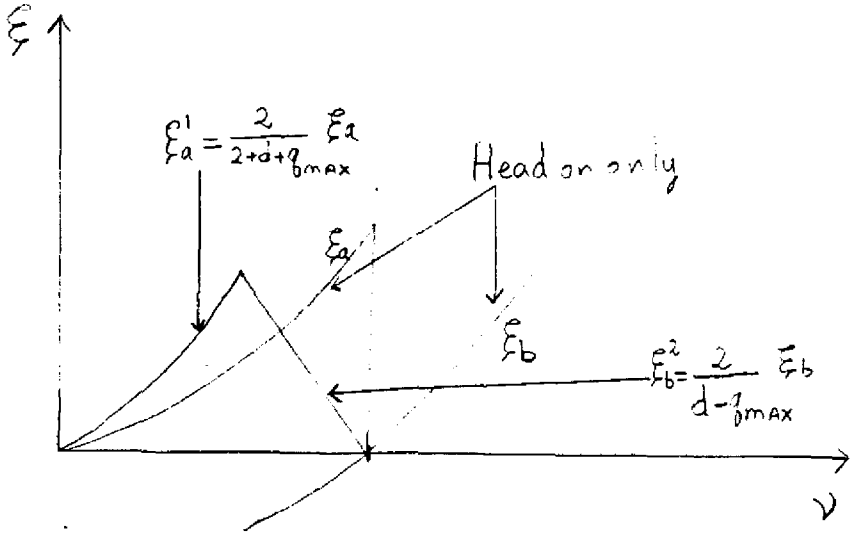
Equation (5.12) gives us a qualitative answer to the long range problem. In the cases we have examined, the coefficient p of (5.1) is negative. This implies that d and q are negative. Furthermore, we see from (5.6a) that q has a minimum at $\lambda=0$:

$$q_{min} = q(0) = d \tag{5.13a}$$

Obviously, there exists an absolute maximum located at some value λ_{max}:

$$q_{max} = q(\lambda_{max}) \leq |d| \quad . \tag{5.12d}$$

Examining (5.12), we conclude that $\xi^1_{\pm}$ is a small perturbation on $\xi\pm$ provided that d is less than 2. More precisely $d+q_{max}$ must be less than 2. On the other hand $\xi^2_{\pm}$ is non-analytic in the long range parameters. Schematically, the effect of these modes is shown on the following figure:



(5.14)

One purpose of this exercise is to study the effect of the crossing angle at the I.P. In this model an increase of the crossing angle amounts to a more negative p. To study this effect, we define the two quantities η and δ:

$$\begin{aligned} q_{max} &= \eta d \\ p_\delta &= \delta p \\ q_\delta &= \delta q \\ d_\delta &= \delta d \end{aligned} \tag{5.15}$$

We need to study the behavior of ξ^1 and ξ^2 under an increase of $\delta(0 \leq \delta \leq \infty)$. Obviously the original mode disappears if $2+\delta d+\eta\delta d$ vanishes:

$$\delta_{transition} = \frac{-2}{(1+\eta)d} \tag{5.16}$$

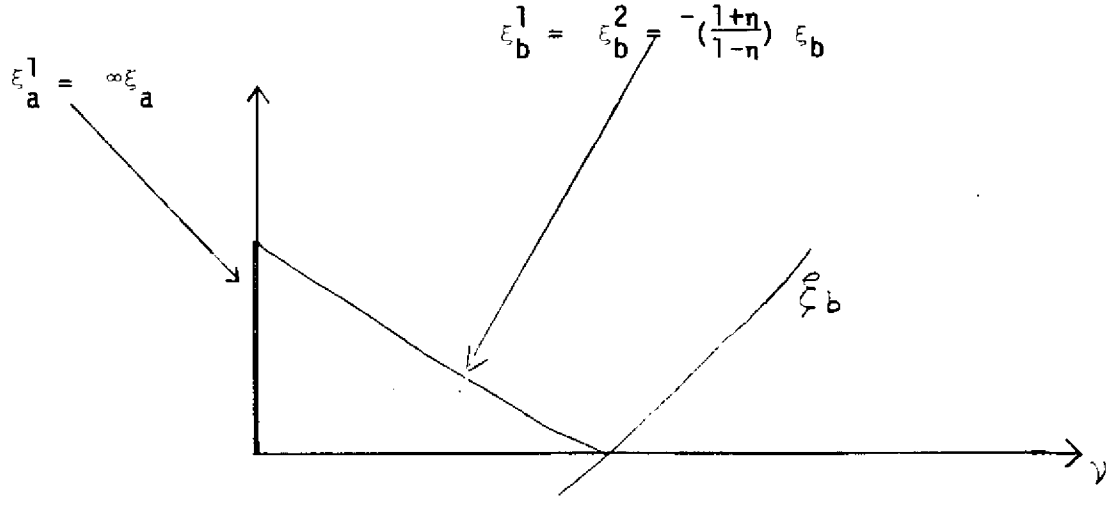
At that particular value of δ, ξ^2 has the value:

$$\xi^2 = \frac{-(1+\eta)}{(1-\eta)}\ \xi\ , \tag{5.17a}$$

In addition ξ^1 evaluated af q=d has the same value:

$$\xi^1 = \frac{2}{2+d\delta+d\delta}\ \xi = -\left(\frac{1+\eta}{1-\eta}\right)\ \xi\ . \tag{5.17b}$$

At the transition given by (5.16), the stability diagram looks as follows:



(5.18)

If δ is larger than $-2/(1+\eta)d$, the stability curve is given by ξ_b^1 evaluated at q=d.

$$\delta > \delta_{transition} \qquad \xi_b^1 = \frac{1}{1+d}\ \xi_b \ . \tag{5.19}$$

6. Study of the H^J with approximate drifts for ρ_ℓ and σ_ℓ.

In the this section, we will derive an approximation for H^J which retains the drift nature of ρ_ℓ and σ_ℓ. We start with $H_{jj+\ell}$, more precisely we will expand the first term in (3.8b):

$$(\rho_\ell x_j - \sigma_\ell^{-1} y_{j+\ell})^2 = (x_j + \frac{\ell L}{2} x_j' - y_{j+\ell} + \frac{\ell L}{2} y_{j+\ell}')^2$$

$$= (\frac{\ell L}{2})^2 (x_j' + y_{j+\ell}')^2 + \ldots \tag{6.1}$$

In equation (6.1) we kept the highest order in L. At the interaction point x, x', y, y' are nearly equal since β^* is one. Therefore the terms we neglected in (6.1) are $\ell L/2$ smaller than the main contribution. This number is at least 7.5 if L=15m as in our simulations.

With this approximation, H^J becomes:

$$H^J = H^0 + \frac{\epsilon p}{2} \left\{ \sum_{\substack{\ell=1 \\ j=0}}^{J,m-1} [(x_j' + y_{j+\ell}')^2 + (y_j' + x_{j+\ell}')^2] \right\} \tag{6.2}$$

Unfortunately, $\exp(:H^J:)$ is not a kick! But since we are working to order ϵ, we can rewrite $\exp(:H^J:)$ as product of two kicks:

$$M = \exp(:H_0:)\exp(:\frac{\epsilon p}{2} \left\{ \sum_{\substack{\ell=1 \\ j=0}}^{J,m-1} (x_j' + y_{j+\ell}')^2 + (y_j' + x_{j+\ell}')^2 \right\} :) = M_0 M_J. \tag{6.3}$$

The matrix representation of M_0 and M_J can be blocked diagonalized using the transformations (5.5) and (5.7). The matrix for M_0 is already known and is unchanged by (5.5) and (5.7):

$$M_0 \Rightarrow \begin{pmatrix} a^0 & b^0 \\ b^0 & a^0 \end{pmatrix} , \tag{6.4}$$

$$a^0 = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix} , \qquad b^0 = \begin{pmatrix} 0 & 0 \\ -\varepsilon & 0 \end{pmatrix} .$$

The effect of M_J can be computed following the steps between (5.4) and (5.9). The result is:

$$M_J \Rightarrow \begin{pmatrix} a_\lambda^J & b_\lambda^J \\ b_\lambda^J & a_\lambda^J \end{pmatrix} \qquad q = 2p \sum_{\ell=1}^{J} \cos\left(\frac{2\pi}{m} \lambda \ell\right) \tag{6.5}$$

$$a_\lambda^J = \begin{pmatrix} 1 & -d\varepsilon \\ 0 & 1 \end{pmatrix} \qquad b_\lambda^J = \begin{pmatrix} 0 & -q\varepsilon \\ 0 & 0 \end{pmatrix} .$$

Finally, we can construct the matrix for M as the product of (6.5) by (6.4). (Lie operators are in the opposite order from their matrix representations. This detail is irrelevant since M_0 and M_J commute to order ε. Furthermore, the characteristic polynomial of $M_0 M_J$ is identical to the polynomial of $M_J M_0$.)

$$M_0 M_J \Rightarrow \begin{pmatrix} a_\lambda & b_\lambda \\ b_\lambda & a_\lambda \end{pmatrix} \tag{6.6}$$

$$a_\lambda = a^J a^0 + b^J b^0 = \begin{pmatrix} 1-\varepsilon^2(d-q) & -d\varepsilon \\ \varepsilon & 1 \end{pmatrix}$$

$$b_\lambda = a^J b^0 + b^J a^0 = \begin{pmatrix} \varepsilon^2(d-q) & -d\varepsilon \\ -\varepsilon & 0 \end{pmatrix} .$$

Using a_λ and b_λ, we can again derive α_μ^λ and β_μ^λ:

$$\alpha_\mu^\lambda = \alpha_\mu^0 - 8c_\mu s_\nu d\varepsilon + 8\, c_\mu c_\nu (d-q)\varepsilon^2 \tag{6.7a}$$

$$\beta_\mu^\lambda = \beta_\mu^0 + 8c_\nu s_\nu d\varepsilon + \Big\{4s_\nu^2(d-q) + 2s_\nu^2(d^2-q^2) + 8\, c_\mu^2 q$$

$$- 8c_\nu^2 d\Big\}\varepsilon^2 + 4\, c_\nu s_\nu (q^2-d^2)\varepsilon^3 \quad . \tag{6.7b}$$

A fancy cubic equation for ε can be written using (6.7) and the stability condition (4.6):

$$8(c_\nu \bar{+} c_\mu)^2 + 8(1+d)(c_\nu \bar{+} c_\mu)s_\nu\varepsilon$$

$$+ \Big\{\pm 8\, c_\mu c_\nu (d-q) + 4\, s_\nu^2(d-q) + 2s_\nu^2(d^2-q^2) + 8c_\mu^2 q - 8\, c_\nu^2 d\Big\}\varepsilon^2$$

$$+ 4c_\nu s_\nu (q^2-d^2)\,\varepsilon^3 = 0 \tag{6.8}$$

Equation (6.8) is more complex than equations (5.11). To understand it, we will look at the limit of small long range effects. Neglecting all terms which are second order in d or q, we find the following result:

$$\xi_\pm^1 = r_a\ \xi_\pm^a \qquad\qquad \xi_\pm^2 = r_b\ \ \xi_\pm^b \tag{6.9a}$$

$$r_a = 1-d + \Big\{(d-q)(2s_\nu^2 \pm 4\, c_\mu c_\nu) + 4\, c_\mu^2 q - 4\, c_\nu^2 d\Big\}\frac{1}{4s_\nu^2} + \ldots \tag{6.9b}$$

$$r_b = \frac{1}{r_a + d - 1} + \cdots \tag{6.9c}$$

Again, we see that r_a and r_b will reach extrema when $q(\lambda)$ reaches an extremum. Analyzing r_a around the point $c\mu \mp c\upsilon = 0$, we get the result:

$$0 \le \upsilon \le \frac{1}{N} \qquad r_a = r_a(q=q_{min}=d)=1 \tag{6.10a}$$

$$\frac{1}{N} \le \upsilon \le .5 \qquad r_a = r_a(q=q_{max}) \tag{6.10b}$$

$$0 \le \upsilon \le .5 - \frac{1}{N} \qquad r_b = r_b(q=q_{max}) \tag{6.10c}$$

$$.5-\frac{1}{N} \le \upsilon \le .5 \qquad r_b = r_b(q=d)=1/d \tag{6.10d}$$

The results of (6.10) are exact even if full cubic is considered. Notice that some branches have a ratio r_a which is tune dependent. This we will see, agrees with the simulation. In fact, the solution of the full cubic, which is analytically possible, will agree very well with the simulation.

Now, we examine the other limit, namely large d and q. Mathematically, the equation for ε becomes:

$$8(c\upsilon+\bar{c}\mu)^2 + 8d(c\upsilon+\bar{c}\mu)s\upsilon\varepsilon + 2s\upsilon^2(d^2-q^2)\varepsilon^2 = 0 \tag{6.11}$$

Equation (6.11) admits two solutions:

$$\xi^1_{\pm} = \frac{1}{d+q} \quad \xi_{\pm} \tag{6.12a}$$

or,

$$\xi^2_{\pm} = \frac{2}{d-q} \quad \xi_{\pm} \ . \tag{6.12b}$$

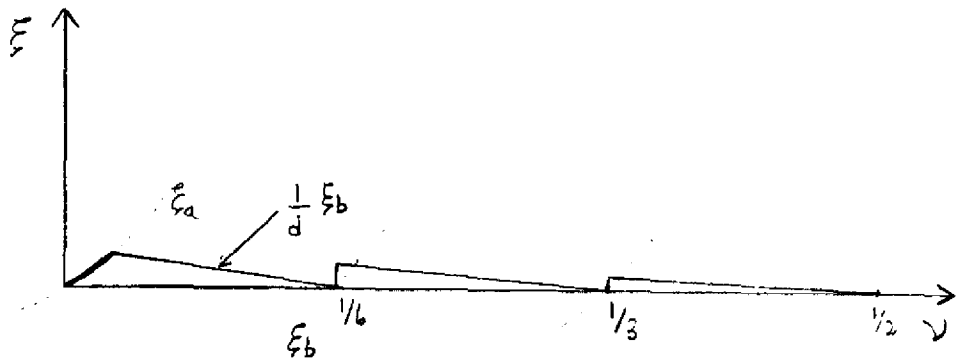
The curve which limits the stability is given by (6.12a) at q=d:

$$\xi^1_{\pm} = \frac{1}{d} \ \xi_b \tag{6.13}$$

The result agrees with (5.19) in the limit of large d.

The initial ξ_a branch disappears everywhere except between ν=0 and ν=1/N. As we know the exact solution of the cubic leads to r_a=1.

The next figure gives a typical large d stability diagram for a six I.P. ring (N=6).



(6.14)

This concludes the analytical treatment of the problem. In the next section, we compare its prediction with the actual simulations.

7. The comparison of the simulations and the analytical results.

The results we will present correspond to two values of the angle between the beams. We ran the 50 μrad and the 10 μrad separation.

In all our simulations, we didn't have any gaps between the trains of particles. The interbunch distance was always 15 meters. Of course, due to computer limitation, we restricted ourselves to a finite number of particles. As a result the machines we modeled were between 0.5 and 3km in circumference. On the other hand, in the analytic computations, we did assume that the function $q(\lambda)$ is continuous. This amounts to an infinite number of bunches and therefore a machine of infinite size!

As the reader will see, the stability curve was fairly insensitive to the number of bunches. In fact, 36 bunches were infinite enough!

The following table gives the parameters of each figure.

Figure	J	M_i=J+1	$M=m_a+m_i$	N	angle(μrad)
1	2	3	6	6	50
2	2	3	6	6	10
3	4	5	6	6	50
4	4	5	6	6	10
5	12	13	33	6	50
6	Plot of $q(\lambda)/d$ for J = 2, 4, 12.				

Each figure displays the simulation result and the curves computed from the drift model of section 5.

We see that the agreement improves as J increases. This is not surprising since the approximation of line (6.1) improves with the distance from the I.P.

Suppose we write $d(\alpha)$ as follows:

$$d(\alpha) = \delta_\alpha \, d(50\mu\text{rad}) \tag{7.1}$$

$$\delta_\alpha = \left(\frac{50}{\alpha}\right)^2$$

We now compute the quantity η of (5.15) using the plots of $q(\lambda)$ given in Fig. 6. Using this value we can compute $\delta_{\text{transition}}$ in the telescopic approximation.

$$J = 2, 4, 12 \qquad \eta = -0.56, -0.38, -0.27$$

$$\delta_{\text{transition}} = 29, 10, 3 \qquad \delta_{10} = 25, \ \delta_{50} = 1 \tag{7.2}$$

According to (7.2), all the 50 μrad cases are under the transition. (Figures 1, 3, and 5).

The (J = 2; α = 10 μrad) case is just under the transition value (δ_{10} = 25 instead of 29). The (J = 4, 12; α = 10 μrad) are both above transition.

These qualitative results agree with the simulation.

8. Conclusion

We found very good agreement between the simulations and the analytical computations. The entire apparatus of group theory can be used to study the higher modes of oscillations. It is our belief that the theory and the approximations used in this paper will be useful in the context of the Vlasov equation.

In fact, the Vlasov equation can be cast in a map formalism identical to the one used here.

J = 2 $\alpha = 50\ \mu rad$

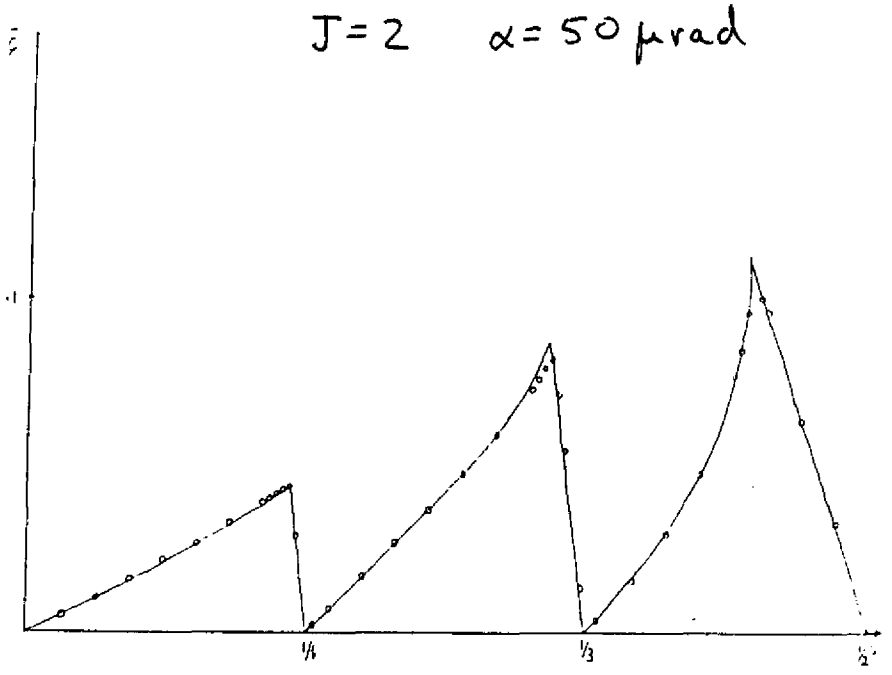


Figure 1

J = 2 $\alpha = 10\ \mu rad$

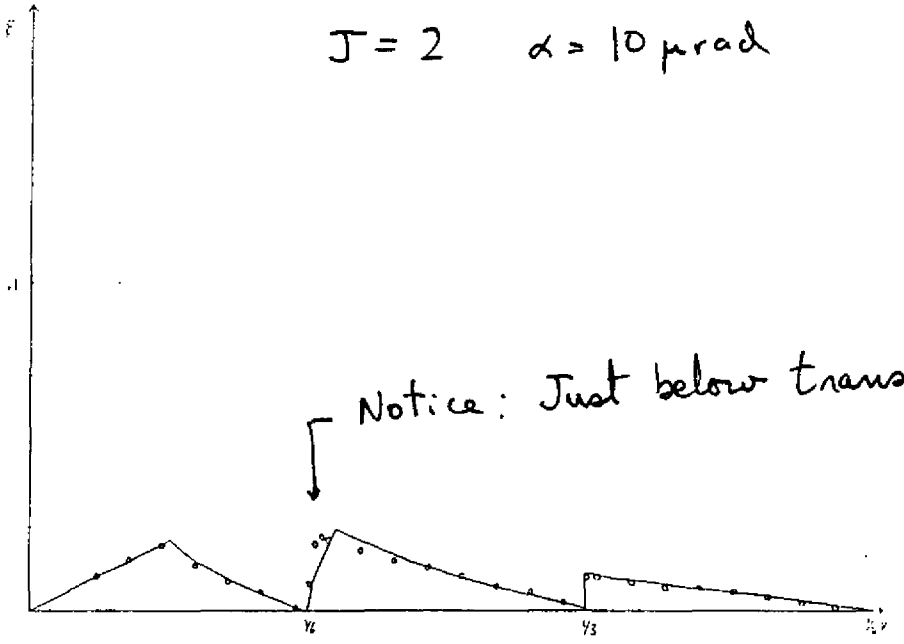


Figure 2

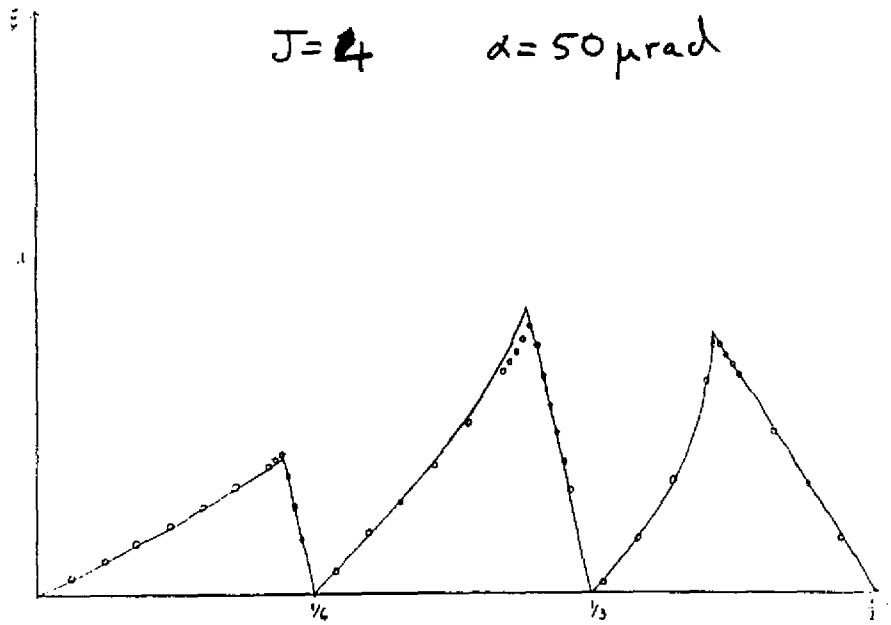


Figure 3

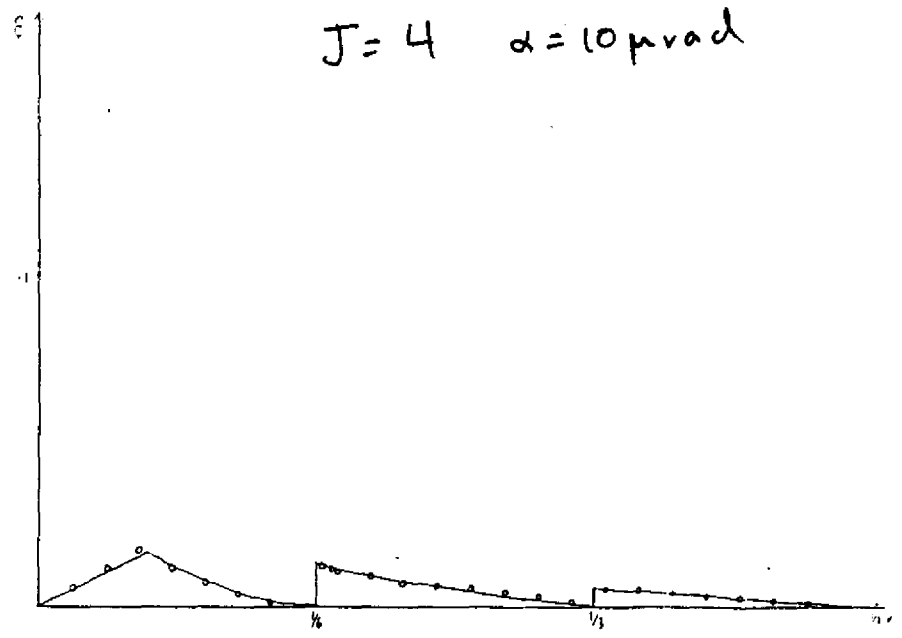


Figure 4

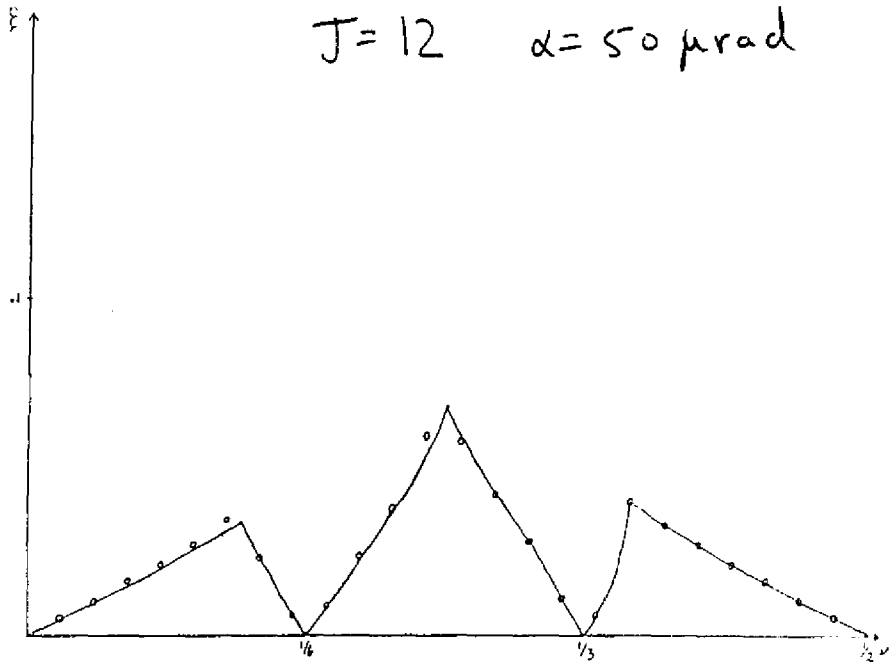


Figure 5

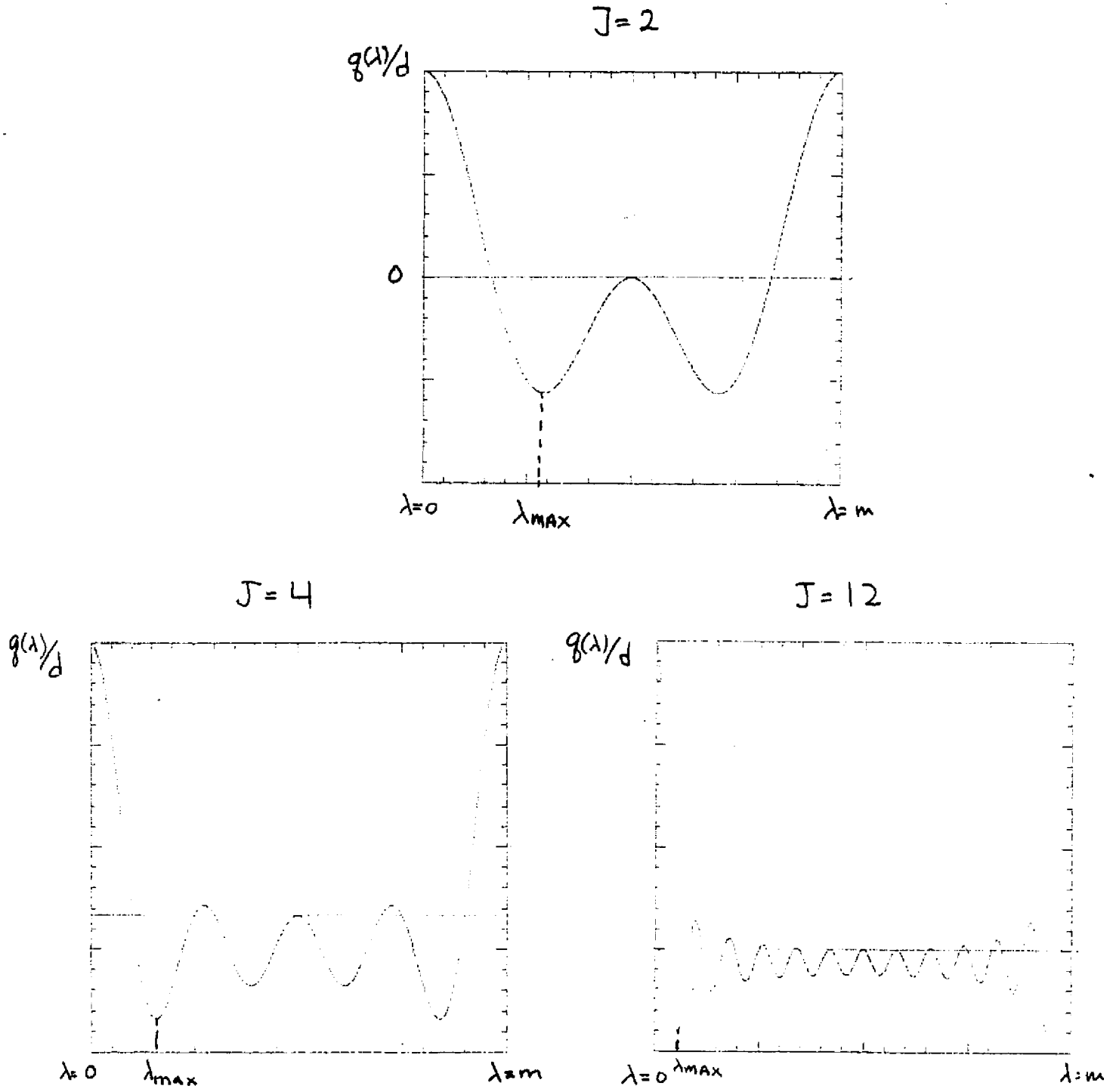


Figure 6