The Møller Luminosity Factor*

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(Dated: September 24, 2003)

Consider two bunches of particles with densities $\rho_1(\mathbf{x},t)$ and $\rho_2(\mathbf{x},t)$ normalized such that

$$\int d^3\mathbf{x}\,\rho_i(\mathbf{x},t) = N_i\,, \quad i=1,2 \tag{1}$$

where N_i is the number of particles in bunch i. Within bunch i, all particles are assumed to move with the common velocity $\mathbf{v}_i$ in the Lab frame. If, during any time interval, the densities ρ_1 and ρ_2 overlap, the bunches will collide with a luminosity given by $\mathcal{L}_{\rm sc} = \int dt d^3\mathbf{x}\,S$, where the luminosity density S is given by [1]

$$S = \rho_1\rho_2\sqrt{(\mathbf{v}_1-\mathbf{v}_2)^2 - \frac{(\mathbf{v}_1\times\mathbf{v}_2)^2}{c^2}}\,. \tag{2}$$

Here $\mathcal{L}_{\rm sc}$, with dimensions of 1/area, is the single-collision luminosity, as emphasized by the subscript "sc". Thus, if σ is a reaction cross section, the dimensionless number $\sigma\mathcal{L}_{\rm sc}$ is the average number of events generated during the collision for this particular cross section. For the case of two trains of colliding bunches, the conventional luminosity $\mathcal{L}$ is obtained by multiplying $\mathcal{L}_{\rm sc}$ by the bunch collision frequency, and the number of events generated per unit time is $\sigma\mathcal{L}$.

If the bunches move collinearly, then $\mathbf{v}_1\times\mathbf{v}_2 = 0$ hence $S = \rho_1\rho_2|\mathbf{v}_1-\mathbf{v}_2|$ which is the intuitively correct result, $|\mathbf{v}_1-\mathbf{v}_2|$ being the relative speed of the bunches. The same result obtains in the nonrelativistic limit even if the bunches do not move collinearly.

Assuming that the particles have mass, a justification of Eq. (2), which is somewhat different than the one in Ref. 1, is obtained by resorting to the basic expression of the luminosity, which is defined in the rest frame of one of the bunches. Thus, in the rest frame of bunch 1, the density is given by $S = \rho_1\rho_2 v_2$, where v_2 is the speed of bunch 2 seen by bunch 1. In the nonrelativistic limit ($v_1, v_2 \ll c$), the immediate generalization of this formula to any reference frame is $S = \rho_1\rho_2|\mathbf{v}_1-\mathbf{v}_2|$ because both the particle density ρ and the relative speed $|\mathbf{v}_1-\mathbf{v}_2|$ are Galileian invariants, hence so is S. In the relativistic case, however, neither ρ nor $|\mathbf{v}_1-\mathbf{v}_2|$ is a Lorentz invariant, hence this expression *can not*, in general, be correct because $|\mathbf{v}_1-\mathbf{v}_2|$ is the relative speed of the bunches only *in the Lab frame*. The $\mathbf{v}_1\times\mathbf{v}_2$ term in (2) is the relativistic correction that makes S a relativistic invariant.

To prove this, we resort again to the basic expression for S by going to the rest frame of one of the bunches. In this frame S is given, by definition, by

$$S = \rho_1\rho_2 v_{12} \tag{3}$$

where v_{12} is the relative speed of the bunches (if, say bunch 1 is at rest, then $v_{12} = v_2$). The general expression for S can be obtained by a Lorentz transformation of this formula to the Lab frame. Since S is a Lorentz invariant, this is most easily achieved by expressing (3) in a manifestly invariant form. Now the relative speed v_{12} is, by definition, a Lorentz scalar; its manifestly invariant form is [2, Eq. (3-4), p. 29][1]

$$v_{12} = \frac{c\sqrt{(u_1u_2)^2 - c^4}}{(u_1u_2)} \tag{4}$$

where u is the four-velocity $u^\mu = \gamma(c,\mathbf{v})$, related to the four-momentum p^μ by $p^\mu = mcu^\mu$, m being the rest-mass of the particle, and where (u_1u_2) denotes the Lorentz scalar product $(u_1u_2) = u_{1\mu}u_2^\mu$. As for the product $\rho_1\rho_2$, one can express it in an invariant form by noting that the particle density ρ transforms as the time component of a Lorentz vector, namely the particle current density $j^\mu = (c\rho, \mathbf{v}\rho)$. Given that the only four-vectors at our disposal are j_i^μ and u_i^μ for $i=1,2$, and that $u^\mu = (\gamma/\rho)j^\mu$, it follows that the only three independent scalars are (j_1j_2), j_1^2 and j_2^2. Of these, only $(j_1j_2)/c^2 = \rho_1\rho_2(1-\mathbf{v}_1\cdot\mathbf{v}_2/c^2)$ reduces to $\rho_1\rho_2$ when either $\mathbf{v}_1=0$ or $\mathbf{v}_2=0$. Therefore, the only acceptable Lorentz invariant generalization of $\rho_1\rho_2$ is

$$\rho_1\rho_2 \to \frac{(j_1j_2)}{c^2} \tag{5}$$

hence the general expression for S is

$$S = \frac{(j_1j_2)\sqrt{(u_1u_2)^2 - c^4}}{c(u_1u_2)}\,. \tag{6}$$

*Work supported by the US DOE under contract DE-AC03-76SF00098.
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[1] Ref. 2 uses natural units ($c=1$); in this note we have restored the c's.

Substituting $(u_1 u_2) = \gamma_1\gamma_2(c^2 - \mathbf{v}_1 \cdot \mathbf{v}_2)$ into (4) yields

$$v_{12} = \frac{c\sqrt{(c^2 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2 - (c^2 - \mathbf{v}_1^2)(c^2 - \mathbf{v}_2^2)}}{c^2 - \mathbf{v}_1 \cdot \mathbf{v}_2} = \frac{\sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2/c^2}}{1 - \mathbf{v}_1 \cdot \mathbf{v}_2/c^2} \qquad (7)$$

where we used the identity $(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 = \mathbf{v}_1^2\mathbf{v}_2^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2$. Expression (7) has the correct properties of a relative speed: it is symmetric under the exchange $1 \leftrightarrow 2$; it reduces to v_2 if $\mathbf{v}_1 = 0$, to v_1 if $\mathbf{v}_2 = 0$, and to $|\mathbf{v}_1 - \mathbf{v}_2|$ in the nonrelativistic limit; it has a maximum value of c when $\mathbf{v}_1 = -\mathbf{v}_2$ with $|\mathbf{v}_1| = |\mathbf{v}_2| = c$; and it vanishes only when $\mathbf{v}_1 = \mathbf{v}_2$.

Combining the above results, Eq. (6) yields Eq. (2), completing the proof.

Equation (6) is a manifestly invariant form of S. An alternative invariant expression follows from substituting $u^\mu = (\gamma/\rho)j^\mu$ and $j^2 = (c\rho/\gamma)^2$ in (6), yielding [3]

$$S = \frac{1}{c}\sqrt{(j_1 j_2)^2 - j_1^2 j_2^2}\,. \qquad (8)$$

For relativistic beams colliding almost head-on, i.e., $v_{12} \simeq c$, or, more precisely, $\gamma_1\gamma_2(1 - \mathbf{v}_1 \cdot \mathbf{v}_2/c^2) \gg 1$, one obtains the approximate expression

$$S \simeq \frac{(j_1 j_2)}{c} = \frac{\rho_1\rho_2}{c}(c^2 - \mathbf{v}_1 \cdot \mathbf{v}_2) \qquad (9)$$

which simplifies to $S \simeq 2c\rho_1\rho_2$ for exactly head-on collisions.

For beams of massless particles the above derivation of S is not possible, as it relies on the rest frame of the bunch as a starting point. However, since expression (2) (or (6) or (8)) for S does not depend explicitly on the particle masses, it is still the valid expression for the luminosity density. For example, if both beams are made up of massless particles, Eq. (8) yields

$$S = \frac{(j_1 j_2)}{c} = c\rho_1\rho_2(1 + \cos\theta)\,, \qquad (10)$$

where θ is the collision crossing angle (i.e., the complement of the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, so that $\cos\theta = +1$ for exactly head-on collisions).

I am indebted to Don Edwards for his keen interest in the Møller factor.

[1] C. Møller, "General Properties of the Characteristic Matrix in the Theory of Elementary Particles. I," K. Danske Vidensk. Selsk. Mat.-Fys. Medd. **23**, 1 (1945).

[2] R. Hagedorn, *Relativistic Kinematics*, W. A. Benjamin, Inc., 1973.

[3] O. Napoly, "The Luminosity for Beam Distributions with Error and Wakefield Effects in Linear Colliders," Part. Accel. **40**, 181 (1993).