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On Radiation by Electrons in a Betatron*

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Transcription notes

Julian Schwinger produced this paper in preprint form in 1945 and, apparently, distributed it only to a few selected colleagues at the time. He later presented the results as a 15-minute invited paper in 1946, at an American Physical Society meeting, under the title "Electron Radiation in High Energy Accelerators" (the abstract is published in Phys. Rev. **70**, 798 (1946)). Although he published this work four years later in revised form ("On the Classical Radiation of Accelerated Electrons," Phys. Rev. **75**, 1912 (1949)), this original version seems fresher and, in some respects, superior to the published one, hence my motivation to make it widely available. For example, the discussion of coherent radiation (shielded and unshielded) included in this version was wholly omitted in the published paper. In addition, this version exhibits many explicit calculations that are of pedagogical value even today for students of synchrotron radiation. But perhaps the most interesting aspect of this paper is that it shows so well the author's superb dexterity in manipulating mathematical expressions to obtain physical conclusions with clarity and efficiency.

In typestting this paper I took the liberty of slightly editing it in several ways: I incorporated into its body two sections that were added by the author after completion of the initial work. As a result, the equation numbering is different from the manuscript, since a few equations that were originally unnumbered but referred to in the text had to be numbered. In the process I deleted, for good measure, the labels of those equations that were not referred to in the text. I corrected a few obvious typographical, grammatical and numerical errors, and added a footnote whenever the difference with the original seemed more than trivial. For this reason I collected the references, which appeared as footnotes in the manuscript, in a separate section at the end of the paper. I substituted the modern conventional abbreviations for units, such as "m" for "meter," "μA" for "microampere," etc. In one sequence of equations I replaced the symbol $d\tau$ by $d^3\mathbf{r}$ for the volume element of certain integrals since, elsewhere in the paper, the author followed the traditional convention, which I respected, of using τ for the proper time of a relativistic particle. By the same token, I replaced τ by T_0 in those equations where τ was used to represent the orbit period, and R by $\mathcal{R}$ for the radiation resistance, since R was used throughout the paper to represent the orbit radius. Finally, I incorporated into the text a few minor equations that were displayed in the original, and replaced stacked fractions by in-line fractions whenever the font size turned out too small to be readable.

By making this magnificent paper widely available I have tried to pay a small tribute to the memory of Julian Schwinger, the master teacher. I am most grateful to Mrs. Clarice Schwinger for kindly granting her approval of this transcription. I am indebted to Professor R. Talman for first showing me the manuscript and for encouraging me to transcribe it, to Professor K. Milton for encouragement, for carefully proofreading the transcription, and for deciding to incorporate it as part of the author's collected works. I am grateful to Dr. G. Decker for kindly lending me his copy of the original, to Professor J. D. Jackson for comments and for proofreading, and to J. Johnson for help in typing the text.

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It has recently been pointed out [1] that the radiative loss of energy produced by the accelerated motion of an electron in an induction accelerator, or betatron, sets a theoretical upper limit to the energy obtainable by such a device. However, the idea appears to be prevalent that this calculation for a single electron does not apply to an actual betatron where many electrons are present simultaneously, for, it is argued, the latter situation corresponds to a steady current which, of course, does not radiate. Otherwise expressed, the fields emitted by the electrons at various points of the circular path interfere destructively and thus suppress the radiation. The same objection to the individual action of the electron is raised, with opposite effect, concerning the radiative loss of energy by a "pulse" of electrons, which travel together distributed over a small part of the orbit. Here, it is argued, the radiation fields of the various electrons will interfere constructively and thus produce a loss of energy proportional to the square of the number of electrons, which would be a much more serious barrier to the attainment of high energies.

It is the purpose of this note to investigate in detail the properties of radiation emitted by a single electron moving in a circular orbit and, with the aid of these results, to study the radiation of these electrons in the two situations mentioned above. The quantities of interest are the total rate of radiation, the rate of radiation into each of the frequencies generated by the electron, and the angular distribution of the radiation emitted at each of these frequencies. Three different methods will be employed, each yielding most advantageously one of these quantities.

The total rate of radiation by an accelerated electron moving with a speed close to that of light is most conveniently obtained by constructing relativistically invariant equations of motion which include the effect of radiation reaction. These can be obtained by a proper generalization of the well-known non-relativistic equations of motion, which include the Lorentz radiation reaction force, as well as the force arising from an external electromagnetic field,

$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{H}\right) + \frac{2}{3}\frac{e^2}{c^3}\frac{d^2\mathbf{v}}{dt^2} \tag{1}$$

or

$$\frac{d\mathbf{p}}{dt} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{H}\right), \qquad \mathbf{p} = m\mathbf{v} - \frac{2}{3}\frac{e^2}{c^3}\frac{d\mathbf{v}}{dt} \tag{2}$$

where $\mathbf{p}$, the electron momentum, is the sum of its kinetic momentum and an "acceleration momentum" arising from the dissipative part of the electron's proper field. The equation of energy deduced from (1) is

$$\frac{d}{dt}\frac{m\mathbf{v}^2}{2} = e\mathbf{E}\cdot\mathbf{v} + \frac{2}{3}\frac{e^2}{c^3}\mathbf{v}\cdot\frac{d^2\mathbf{v}}{dt^2}$$

or

$$\frac{dE}{dt} = e\mathbf{E}\cdot\mathbf{v} - \frac{2}{3}\frac{e^2}{c^3}\left(\frac{d\mathbf{v}}{dt}\right)^2, \qquad E = \frac{m\mathbf{v}^2}{2} - \frac{2}{3}\frac{e^2}{c^3}\frac{d}{dt}\frac{\mathbf{v}^2}{2}\,. \tag{3}$$

Thus the classical Larmor formula describes the rate of radiative dissipation of an energy, E, which consists of the kinetic energy and an "acceleration energy." The relativistic generalization of Eqs. (2) and (3) is

$$\frac{dp_\mu}{ds} = \frac{e}{c}\sum_{\nu=1}^{4}F_{\mu\nu}\frac{dx_\nu}{ds} - \frac{2}{3}\frac{e^2}{c}\frac{dx_\mu}{ds}\sum_{\nu=1}^{4}\left(\frac{d^2x_\nu}{ds^2}\right)^2, \qquad p_\mu = mc\frac{dx_\mu}{ds} - \frac{2}{3}\frac{e^2}{c}\frac{d^2x_\mu}{ds^2} \tag{4}$$

where the four-vector p_μ contains the energy and momentum,

$$p_\mu = (\mathbf{p}, iE/c),$$

the four-vector of position is

$$x_\mu = (\mathbf{r}, ict),$$

the differential of proper time, ds, is defined by

$$ds^2 = -\sum_{\mu=1}^{4}(dx_\mu)^2 = c^2dt^2 - d\mathbf{r}^2,$$

and $F_{\mu\nu}$ is the six-vector of the external field,

$$F_{12} = H_z, \quad F_{14} = -iE_x, \quad \text{etc.}$$

It is easily verified that (4) reduces to (2) and (3) in a coordinate system with respect to which the electron is instantaneously at rest ($v/c \ll 1$), and, being a four-vector equation, its validity in all coordinate systems is established. On eliminating p_μ, we obtain the relativistic generalization of Eq. (1),

$$mc^2 \frac{d^2 x_\mu}{ds^2} = e \sum_\nu F_{\mu\nu} \frac{dx_\nu}{ds} + \frac{2}{3} e^2 \left[\frac{d^3 x_\mu}{ds^3} - \frac{dx_\mu}{ds} \sum_\nu \left(\frac{d^2 x_\nu}{ds^2} \right)^2 \right] = e \sum_\nu \left(F_{\mu\nu} + \tilde{f}_{\mu\nu} \right) \frac{dx_\nu}{ds} \tag{5}$$

where

$$\tilde{f}_{\mu\nu} = -\frac{2}{3} e \left(\frac{d^3 x_\mu}{ds^3} \frac{dx_\nu}{ds} - \frac{d^3 x_\nu}{ds^3} \frac{dx_\mu}{ds} \right)$$

is the six-vector of the dissipative part of the electron's self-field (half the difference between the retarded and the advanced field of the point charge). These are the classical equations of motion proposed by Dirac [2]. The fourth component of Eq. (4), in three-vector notation, reads

$$\frac{dE}{dt} = e\mathbf{E} \cdot \mathbf{v} - \frac{2}{3} \frac{e^2}{m^2 c^3} \left(\frac{E}{mc^2} \right)^2 \left[\left(\frac{d\mathbf{p}}{dt} \right)^2 - \frac{1}{c^2} \left(\frac{dE}{dt} \right)^2 \right] \tag{6}$$

in which we have disregarded the small difference between the total and the kinetic energy and momentum (which is justified whenever the classical theory is applicable). We thus obtain the relativistic generalization of the Larmor formula, *viz.*

$$-\left(\frac{dE}{dt} \right)_{\text{rad.}} = \frac{2}{3} \frac{e^2}{m^2 c^3} \left(\frac{E}{mc^2} \right)^2 \left[\left(\frac{d\mathbf{p}}{dt} \right)^2 - \frac{1}{c^2} \left(\frac{dE}{dt} \right)^2 \right]. \tag{7}$$

To apply this result to the betatron, we consider an electron moving in a circular orbit of radius R under the influence of a slowly-varying magnetic field $H(t)$. Note that $(dE/cdt)^2$ is thoroughly negligible in comparison with $(d\mathbf{p}/dt)^2$, for

$$\left(\frac{d\mathbf{p}}{dt} \right)^2 \sim \frac{\mathbf{p}^2}{T_0^2}, \qquad \left(\frac{1}{c} \frac{dE}{dt} \right)^2 \sim \frac{(E/c)^2}{T^2} \sim \frac{\mathbf{p}^2}{T^2}$$

where T_0 is the required time to traverse the orbit, and T is the time required to build up the magnetic field to its maximum value. Thus the first term is larger than the second in the ratio $(T/T_0)^2$, or the square of the total number of revolutions made by an electron in acquiring the final energy, an extremely large number. Since the radiation reaction force is small in comparison with that due to the magnetic field, we write

$$-\left(\frac{dE}{dt} \right)_{\text{rad.}} = \frac{2}{3} c \left(\frac{e^2}{mc^2} \right)^2 \left(\frac{E}{mc^2} \right)^2 \left| \frac{\mathbf{v}}{c} \times \mathbf{H} \right|^2 \tag{8}$$

which is the formula of Iwanenko and Pomeranchuk. However, it is more convenient to introduce the angular velocity of the electron

$$\omega = \frac{v}{R} = \frac{c\beta}{R}$$

and write

$$\left(\frac{d\mathbf{p}}{dt} \right)^2 = \frac{\omega}{R} \frac{\beta^3}{c} E^2$$

whence

$$-\left(\frac{dE}{dt} \right)_{\text{rad.}} = \frac{2}{3} \frac{\omega e^2}{R} \left(\frac{E}{mc^2} \right)^4, \qquad \frac{E}{mc^2} \gg 1. \tag{9}$$

Thus the energy lost per revolution is given by the simple result

$$(\delta E)_{\text{rad.}} = \frac{4\pi}{3}\frac{e^2}{R}\left(\frac{E}{mc^2}\right)^4$$

which, for an energy of 10^8 eV $\simeq 200\ mc^2$, and a radius of 0.5 m, is ~ 20 eV. The total energy radiated is roughly equal to this result multiplied by the total number of revolutions required to reach the final energy, $\omega T/2\pi$, which, for a radius of 0.5 m and $T = 5 \times 10^{-3}$ s, is $\sim 5 \times 10^5$, yielding a total energy loss $(\Delta E)_{\text{rad.}} \sim 10^7$ eV, which is, of course, an over-estimate. For a more precise calculation we recall the the momentum, and to a good approximation in the relativistic region, the energy, is proportional to the magnetic field. If the latter increases sinusoidally to its maximum value in the time T, the time variation of the energy is

$$E = E_0 \sin\frac{\pi t}{2T}$$

and consequently

$$(\Delta E)_{\text{rad.}} = \frac{2}{3}\frac{\omega e^2}{R}\left(\frac{E_0}{mc^2}\right)^4 \int_0^T dt\, \sin^4\frac{\pi t}{2T} = \frac{1}{4}\omega T\frac{e^2}{R}\left(\frac{E_0}{mc^2}\right)^4 ,$$

which differs from the rough estimate given above by the factor 3/8, the average value of $\sin^4(\pi t/2T)$. The fraction of the final energy lost in the radiation is given by

$$\frac{(\Delta E)_{\text{rad.}}}{E_0} = \frac{\omega T}{4}\frac{e^2/mc^2}{R}\left(\frac{E_0}{mc^2}\right)^3$$

which is $\sim 3\%$ for the conditions $E_0/mc^2 = 200$, $R = 0.5$ m and $T = 5 \times 10^{-3}$ s. Of course, in consequence of the radiative losses, the energy does not increase in proportion to the magnetic field, but this can hardly be taken into account without also considering the defocusing action of the dissipative forces, which will require a more precise solution of the equations of motion (5).

However, the general nature of the orbits is easily seen, for to a high degree of approximation, the conditions for a circular path of radius r remains valid:

$$pc \simeq E = eHr \tag{10}$$

and since the energy does not increase proportionally with the magnetic field, the radius of the orbit gradually shrinks. The effect will eventually reduce the rate at which the electric field transfers energy to the electron, but for small radiation losses this can be ignored, since the electric field is a maximum at the radius of the equilibrium orbit. To a first approximation, then, the electron energy, as a function of time, is given by

$$E(t) = E_0(t) - \Delta E(t) \tag{11}$$

where

$$E_0(t) = E_0 \sin\frac{\pi t}{2T} ,$$

$$\Delta E(t) = \frac{2}{3}\frac{\omega e^2}{R}\left(\frac{E_0}{mc^2}\right)^4 \int_0^t dt \sin^4\frac{\pi t}{2T} \equiv \frac{8}{3}\Delta E \int_0^t \frac{dt}{T}\sin^4\frac{\pi t}{2T}$$

and E_0 may now be defined as the maximum energy of the electron ignoring radiation effects, and ΔE is the total radiation loss in attaining the maximum magnetic field, supposing the orbit radius to be a constant. The time dependence of the orbit radius can be inferred from (10), for a known radial variation of the magnetic field. We shall assume that

$$H(r,t) = H_0(t)\left(\frac{r}{R}\right)^{-n} = H_0 \sin\frac{\pi t}{2T}\left(\frac{r}{R}\right)^{-n}$$

where n is a constant, such that $0 < n < 1$, in order that the orbit be stable. Thus

$$E(t) = eH_0(t)R\left(\frac{r}{R}\right)^{1-n} = E_0(t)\left(\frac{r}{R}\right)^{1-n} = E_0(t)\left(1 - \frac{\delta R}{R}\right)^{1-n}$$

and therefore, to a first approximation,

$$\frac{\delta R}{R} = \frac{1}{1-n}\frac{\Delta E(t)}{E_0(t)}$$

which states that the fractional decrease in radius equals the fractional loss of energy, multiplied by $1/(1-n)$, which is 4, if $n = 3/4$. Hence, when the magnetic field is a maximum ($t = T$),

$$\frac{\delta R}{R} = 4\frac{\Delta E}{E_0} = 3.8\% \tag{12}$$

or a decrease in the radius of $\delta R = 3.2$ cm.[2]

A more exact solution of the equations of motion is possible when the radial dependence of the magnetic field is $r^{-3/4}$, as Blewett has shown. The essential reason for this is that the rate of loss of angular momentum is then only a function of time, thereby greatly facilitating integration. The equation of motion deduced from (5), on treating the radiation reaction force as small compared with that due to the magnetic field, is

$$\frac{d\mathbf{p}}{dt} = -e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{H}\right) - \frac{2}{3}\frac{\mathbf{v}}{c}\left(\frac{e^2}{mc^2}\right)^2\left(\frac{E}{mc^2}\right)^2 H^2 \tag{13}$$

which may be easily verified by comparing the energy equation deduced from it with (6) and (8). Note that the electron charge is written as $-e$, to avoid sign difficulties. With e positive, a magnetic field directed along the positive z axis, the electron circulates in the positive, counterclockwise sense. The time varying magnetic field, and its concomitant electric field, are derived from the single vector potential component

$$A_\varphi = A(r,t), \quad H_z = H(r,t) = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)A(r,t), \quad E_\varphi = -\frac{1}{c}\frac{\partial}{\partial t}A(r,t) \tag{14}$$

The rate of change of the z component of the kinetic angular momentum, as deduced from (13), is

$$\frac{d}{dt}(\mathbf{r}\times\mathbf{p})_z = \frac{e}{c}\left(\frac{\partial}{\partial t}rA + \frac{dr}{dt}\frac{\partial}{\partial r}rA\right) - \frac{2}{3}\left(\mathbf{r}\times\frac{\mathbf{v}}{c}\right)_z\left(\frac{e^2}{mc^2}\right)^2\left(\frac{E}{mc^2}\right)^2 H^2$$

or

$$\frac{d}{dt}\left[r\left(p - \frac{e}{c}A\right)\right] = -\frac{2}{3}\left(\frac{e^2}{mc^2}\right)^2\left(\frac{E}{mc^2}\right)^2 H^2 r \tag{15}$$

where $p \simeq eHr/c$ represents the angular component of the linear momentum, and the angular component of the linear velocity has been approximated by c. Introducing the approximate relation $E = eHr$ on the right side of (15), we find

$$\frac{d}{dt}\left[r\left(p - \frac{e}{c}A\right)\right] = -\frac{2}{3}\left(\frac{e^2}{mc^2}\right)^2\left(\frac{e}{mc^2}\right)^2 H^4 r^3$$

which indeed is independent of r if $H \sim r^{-3/4}$. In this circumstance

$$\frac{d}{dt}\left[r\left(p - \frac{e}{c}A\right)\right] = -\frac{2}{3}\left(\frac{e^2}{mc^2}\right)^2\left(\frac{e}{mc^2}\right)^2 H_0^4 R^3 \sin^4\frac{\pi t}{2T} = -\frac{2}{3}\frac{e^2}{R}\left(\frac{E_0}{mc^2}\right)^4 \sin^4\frac{\pi t}{2T}$$

and

$$r\left(p - \frac{e}{c}A\right) = -\frac{2}{3}\frac{e^2}{R}\left(\frac{E_0}{mc^2}\right)^4\int_0^t dt \sin^4\frac{\pi t}{2T} = -\frac{1}{\omega}\Delta E(t).$$

[2] These numbers are not consistent with those on page 5; the correct values are $\delta R/R = 14\%$ and $\delta R = 7$ cm.

The constant of integration is zero since at $t = 0$, both p and A vanish. In the absence of radiation, $p - eA/c = 0$ for all time and it is this relation combined with (10) and (14) that yields $\partial A/\partial r = 0$ as the condition for the equilibrium orbit. This result has already been mentioned in the statement that the electric field is a maximum on the equilibrium orbit. The effect of radiation on the orbit is now described by

$$\frac{e}{c} r^2 \frac{\partial A}{\partial r} = -\frac{1}{\omega} \Delta E(t).$$

The vector potential can be constructed from the assumed magnetic field, together with the condition that $\partial A/\partial r = 0$ at $r = R$, or equivalently, that $A = rH$ at $r = R$. We find

$$rA(r,t) = R^2 H_0(t) \left[\frac{4}{5} \left(\frac{r}{R} \right)^{5/4} + \frac{1}{5} \right]$$

and

$$r^2 \frac{\partial}{\partial r} A(r,t) = \frac{1}{5} R^2 H_0(t) \left[\left(\frac{r}{R} \right)^{5/4} - 1 \right].$$

Hence

$$\left(\frac{r}{R} \right)^{5/4} = 1 - 5 \frac{\Delta E(t)}{E_0(t)}$$

or

$$\frac{r}{R} = \left(1 - 5 \frac{\Delta E(t)}{E_0(t)} \right)^{4/5}$$

which properly reduces to (12) for small radiation losses. The energy, as a function of time, is then given by

$$E = eHr = E_0(t) \left(\frac{r}{R} \right)^{1/4} = E_0(t) \left(1 - 5 \frac{\Delta E(t)}{E_0(t)} \right)^{1/5}$$

which is in agreement with (11) for small radiation losses.

We may note that the situation is quite different when the acceleration of the electron is in the direction of its velocity, as in a micro-wave linear accelerator. In this case the single component of momentum is practically equal to E/c, and the two terms of Eq. (7) cancel. More exactly,

$$p \simeq \frac{E}{c} - \frac{1}{2c} \frac{(mc^2)^2}{E}, \qquad \frac{dp}{dt} = \frac{1}{c} \frac{dE}{dt} \left[1 + \frac{1}{2} \left(\frac{mc^2}{E} \right)^2 \right]$$

and

$$\left(\frac{d\mathbf{p}}{dt} \right)^2 - \frac{1}{c^2} \left(\frac{dE}{dt} \right)^2 \simeq \frac{1}{c^2} \left(\frac{mc^2}{E} \right)^2 \left(\frac{dE}{dt} \right)^2.$$

Therefore,

$$-\left(\frac{dE}{dt} \right)_{\text{rad.}} = \frac{2}{3} \frac{e^2}{m^2 c^5} \left(\frac{dE}{dt} \right)^2.$$

If, for example, the electron gains energy at a steady rate in the accelerating field, the fraction of the energy lost in attaining the final energy E_0 is

$$\frac{(\Delta E)_{\text{rad.}}}{E_0} = \frac{2}{3} \frac{e^2}{mc^2} \frac{d}{dx} \left(\frac{E}{mc^2} \right),$$

written in terms of the rate of energy gain per unit distance. Hence, in order to lose an appreciable fraction of its final energy in radiation, the accelerating field must supply an energy equal to mc^2 in a distance equal to the classical radius of the electron.

An electron moving in a circular path radiates at frequencies which are integral multiples of the fundamental (angular) frequency ω. Our next task is to determine the spectrum of the radiation, *i.e.*, the power radiated into each of the harmonics. The method employed will be that of evaluating the average rate at which the electron does work on the field. In terms of the cylindrical coordinate system ρ, φ, z, the position of the electron at time t is specified by $\rho = R$, $\varphi = \varphi_0 + \omega t$, $z = 0$, where φ_0 is the angular position at the arbitrary time $t = 0$. The charge density of the point electron can be represented in terms of delta functions

$$\rho(\mathbf{r}, t) = e\delta(\mathbf{r} - \mathbf{r}_e(t)) = e\frac{\delta(\rho - R)}{R}\delta(z)\delta(\varphi - \varphi_0 - \omega t)$$

where $\delta(\varphi)$ is understood as a periodic delta function, that is, φ is always to be reduced, modulo 2π, to the fundamental range, which is chosen to be, say, $-\pi$ to π.[3] This periodic function can be expanded in a Fourier series,

$$\delta(\varphi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\varphi} \tag{16}$$

whence the charge density has the form

$$\rho(\mathbf{r}, t) = \sum_{n=-\infty}^{\infty} e^{-in\omega t}\rho_n(\mathbf{r}) \tag{17}$$

where

$$\rho_n(\mathbf{r}) = e\frac{\delta(\rho - R)}{R}\delta(z)\frac{1}{2\pi}e^{in(\varphi-\varphi_0)}$$

Similarly, the current density

$$\mathbf{J}(\mathbf{r}, t) = \rho(\mathbf{r}, t)\mathbf{v}(t) = \mathbf{e}_\varphi\, v\rho(\mathbf{r}, t)$$

can be written

$$\mathbf{J}(\mathbf{r}, t) = \sum_{n=-\infty}^{\infty} e^{-in\omega t}\mathbf{J}_n(\mathbf{r})$$

with

$$\mathbf{J}_n(\mathbf{r}) = \mathbf{e}_\varphi\, ev\frac{\delta(\rho - R)}{R}\delta(z)\frac{1}{2\pi}e^{in(\varphi-\varphi_0)} \tag{18}$$

where $\mathbf{e}_\varphi$ is a unit vector in the direction of increasing φ, that is, tangential to the orbit.

The vector and scalar potential deduced from the retarded time solution of the field equation also have the form of Fourier series in time. The n-th Fourier amplitudes are, respectively,

$$\mathbf{A}_n(\mathbf{r}) = \frac{1}{c}\int d^3\mathbf{r}'\,\frac{e^{in\omega|\mathbf{r}-\mathbf{r}'|/c}}{|\mathbf{r}-\mathbf{r}'|}\,\mathbf{J}_n(\mathbf{r}') \tag{19a}$$

$$\phi_n(\mathbf{r}) = \int d^3\mathbf{r}'\,\frac{e^{in\omega|\mathbf{r}-\mathbf{r}'|/c}}{|\mathbf{r}-\mathbf{r}'|}\,\rho_n(\mathbf{r}')\,. \tag{19b}$$

In terms of these, the n-th Fourier amplitude of the electric field is

$$\mathbf{E}_n(\mathbf{r}) = i\frac{n\omega}{c}\mathbf{A}_n(\mathbf{r}) - \nabla\phi_n(\mathbf{r}). \tag{20}$$

The average power dissipated by the electron in the form of radiation equals the average rate at which the electron does work on the field,

$$P = -\int d^3\mathbf{r}\,\overline{\mathbf{E}\cdot\mathbf{J}}\,.$$

[3] This sentence was slightly edited for readability.

On inserting the Fourier series for $\mathbf{E}$ and $\mathbf{J}$, and noting, for example, that $\mathbf{E}_{-n} = \mathbf{E}_n^*$, in consequence of the reality of $\mathbf{E}$, we obtain

$$P = -\sum_{n=-\infty}^{\infty} \int d^3\mathbf{r}\, \mathbf{J}_n^* \cdot \mathbf{E}_n = -2\mathrm{Re}\left\{ \sum_{n=1}^{\infty} \int d^3\mathbf{r}\, \mathbf{J}_n^* \cdot \mathbf{E}_n \right\}$$

in which we have discarded the harmonic $n = 0$, which, of course, carries away no energy. We can now identify the n-th terms of this sum as the average power radiated in the n-th harmonic:

$$P_n = -2\mathrm{Re}\left\{ \int d^3\mathbf{r}\, \mathbf{J}_n^* \cdot \mathbf{E}_n \right\}.$$

The latter expression is conveniently re-written on replacing the electric field by its expression in terms of the potentials, which, together with an integration by parts, yields

$$P_n = 2\mathrm{Re}\left\{ in\omega \int d^3\mathbf{r} \left[\rho_n^* \phi_n - \frac{1}{c} \mathbf{J}_n^* \cdot \mathbf{A}_n \right] \right\}$$

on employing the equation of charge conservation, $\nabla \cdot \mathbf{J}_n = in\omega\rho_n$. If we now insert the explicit expressions for the potentials, we obtain

$$P_n = 2\mathrm{Re}\left\{ in\omega \int d^3\mathbf{r}\, d^3\mathbf{r}' \, \frac{e^{in\omega|\mathbf{r}-\mathbf{r}'|/c}}{|\mathbf{r}-\mathbf{r}'|} \left[\rho_n^*(\mathbf{r})\rho_n(\mathbf{r}') - \frac{1}{c^2} \mathbf{J}_n^*(\mathbf{r}) \cdot \mathbf{J}_n(\mathbf{r}') \right] \right\}. \tag{21}$$

Now performing the trivial integrations with respect to ρ and z, this becomes

$$P_n = \mathrm{Re}\left\{ in\frac{\omega e^2}{R} \int \frac{d\varphi}{2\pi} \frac{d\varphi'}{2\pi} \frac{e^{2in\beta|\sin((\varphi-\varphi')/2)|}}{\left|\sin \frac{\varphi-\varphi'}{2}\right|} \left[1 - \beta^2 \cos(\varphi-\varphi')\right] e^{-in(\varphi-\varphi')} \right\}$$

in which we have written $\omega R/c = v/c = \beta$. The introduction of $\varphi - \varphi'$ as a variable further simplifies this to

$$P_n = \mathrm{Re}\left\{ in\frac{\omega e^2}{R} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} \frac{e^{2in\beta|\sin(\varphi/2)|}}{\left|\sin\frac{\varphi}{2}\right|} \left(1 - \beta^2\cos\varphi\right) e^{-in\varphi} \right\}. \tag{22}$$

Note that $e^{-in\varphi}$ can be replaced by $\cos n\varphi$ without[4] altering the value of the integral. On taking the real part of the resultant expression, we get

$$P_n = -n\frac{\omega e^2}{R} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} \frac{\sin\left(2n\beta\sin\frac{\varphi}{2}\right)}{\sin\frac{\varphi}{2}} \left(1-\beta^2\cos\varphi\right)\cos n\varphi\,. \tag{23}$$

Before evaluating this expression, it is well to verify that the total power radiated in all harmonics agrees with that obtained previously. For this purpose, it is convenient to sum n from $-\infty$ to ∞, and we therefore write

$$P = \sum_{n=1}^{\infty} P_n = -\frac{\omega e^2}{2R} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} \frac{1-\beta^2\cos\varphi}{\sin\frac{\varphi}{2}} \sum_{n=-\infty}^{\infty} n\cos n\varphi\, \sin\left(2n\beta\sin\frac{\varphi}{2}\right).$$

Now,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} n\cos n\varphi\, \sin\left(2n\beta\sin\frac{\varphi}{2}\right) &= \left[\frac{d}{d\varphi}\right] \sum_{n=-\infty}^{\infty} \sin n\varphi\, \sin\left(2n\beta\sin\frac{\varphi}{2}\right) \\ &= \frac{1}{2}\left[\frac{d}{d\varphi}\right] \sum_{n=-\infty}^{\infty} \left\{ e^{in(\varphi-2\beta\sin(\varphi/2))} - e^{in(\varphi+2\beta\sin(\varphi/2))} \right\} \end{aligned}$$

[4] The manuscript says “with” instead of “without.”

where $[d/d\varphi]$ signifies that $\sin(\varphi/2)$ is to be regarded as constant in differentiating with respect to φ. We have already remarked (Eq. (16)) that[5]

$$\delta(\varphi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\varphi}, \qquad -\pi < \varphi < \pi$$

and, in consequence,

$$P = \beta\frac{\omega e^2}{R}\int_{-\pi}^{\pi} d\varphi\left(1-\beta^2\cos\varphi\right)\left[\frac{d}{d\varphi}\right]\left\{\frac{\delta\left(\varphi+2\beta\sin\frac{\varphi}{2}\right)-\delta\left(\varphi-2\beta\sin\frac{\varphi}{2}\right)}{4\beta\sin\frac{\varphi}{2}}\right\}. \tag{24}$$

It is now convenient to expand the delta functions $\delta(\varphi \pm 2\beta\sin(\varphi/2))$ in a power series in $2\beta\sin(\varphi/2)$; thus

$$\delta(\varphi \pm 2\beta\sin\frac{\varphi}{2}) = \sum_{n=0}^{\infty}\frac{\left(\pm 2\beta\sin\frac{\varphi}{2}\right)^n}{n!}\left(\frac{d}{d\varphi}\right)^n\delta(\varphi)$$

from which we deduce

$$\left[\frac{d}{d\varphi}\right]\left\{\frac{\delta\left(\varphi+2\beta\sin\frac{\varphi}{2}\right)-\delta\left(\varphi-2\beta\sin\frac{\varphi}{2}\right)}{4\beta\sin\frac{\varphi}{2}}\right\} = \sum_{n=0}^{\infty}\frac{\left(2\beta\sin\frac{\varphi}{2}\right)^{2n}}{(2n+1)!}\left(\frac{d}{d\varphi}\right)^{2n+2}\delta(\varphi)$$

on recalling the significance of the operator $[d/d\varphi]$. A suitable integration by parts transforms Eq. (24) into

$$\begin{aligned} P &= \beta\frac{\omega e^2}{R}\sum_{n=0}^{\infty}\int_{-\pi}^{\pi} d\varphi\,\delta(\varphi)\left(\frac{d}{d\varphi}\right)^{2n+2}\frac{\left(2\beta\sin\frac{\varphi}{2}\right)^{2n}}{(2n+1)!}\left(1-\beta^2\cos\varphi\right) \\ &= \beta\frac{\omega e^2}{R}\left\{(1-\beta^2)\sum_{n=0}^{\infty}\left[\left(\frac{d}{d\varphi}\right)^{2n+2}\frac{\left(2\beta\sin\frac{\varphi}{2}\right)^{2n}}{(2n+1)!}\right]_{\varphi=0}+\frac{1}{2}\sum_{n=0}^{\infty}\left[\left(\frac{d}{d\varphi}\right)^{2n+2}\frac{\left(2\beta\sin\frac{\varphi}{2}\right)^{2n+2}}{(2n+1)!}\right]_{\varphi=0}\right\} \end{aligned}$$

in which the second step involves the fundamental property of the delta function, and the relation

$$1-\beta^2\cos\varphi = 1-\beta^2+2\beta^2\sin^2\frac{\varphi}{2}\,. \tag{25}$$

Performing the indicated operations, we obtain finally

$$\begin{aligned} P &= \beta\frac{\omega e^2}{R}\left[\beta^2\sum_{n=0}^{\infty}(n+1)\beta^{2n}-(1-\beta^2)\frac{1}{6}\sum_{n=0}^{\infty}n(n+1)\beta^{2n}\right] \\ &= \frac{\omega e^2}{R}\beta^3\left[\frac{d}{d\beta^2}\frac{1}{1-\beta^2}-(1-\beta^2)\frac{1}{6}\left(\frac{d}{d\beta^2}\right)^2\frac{1}{1-\beta^2}\right] \\ &= \frac{2}{3}\frac{\omega e^2}{R}\frac{\beta^3}{(1-\beta^2)^2} \\ &= \frac{2}{3}\frac{\omega e^2}{R}\beta^3\left(\frac{E}{mc^2}\right)^4 \end{aligned}$$

in complete agreement with Eq. (9).

[5]In the manuscript the range for φ is declared to be $-2\pi < \varphi < 2\pi$, which is unnecessarily twice as wide as it needs to be.

We return to the formula for the power radiated into the n-th harmonic, Eq. (23), which may be written

$$P_n = -n\frac{\omega e^2}{R}\left[(1-\beta^2)\int\limits_{-\pi}^{\pi}\frac{d\varphi}{2\pi}\frac{\sin\left(2n\beta\sin\frac{\varphi}{2}\right)}{\sin\frac{\varphi}{2}}\cos n\varphi + 2\beta^2\int\limits_{-\pi}^{\pi}\frac{d\varphi}{2\pi}\sin\left(2n\beta\sin\frac{\varphi}{2}\right)\sin\frac{\varphi}{2}\cos n\varphi\right]$$

with the aid of (25). Both integrals can be expressed in terms of Bessel functions, for the operations of integration and differentiation with respect to z, applied to the equation

$$\int\limits_{-\pi}^{\pi}\frac{d\varphi}{2\pi}\cos\left(z\sin\frac{\varphi}{2}\right)\cos n\varphi = J_{2n}(z)$$

yield

$$\int\limits_{-\pi}^{\pi}\frac{d\varphi}{2\pi}\frac{\sin\left(z\sin\frac{\varphi}{2}\right)}{\sin\frac{\varphi}{2}}\cos n\varphi = \int\limits_{0}^{z}dx\,J_{2n}(x)$$

$$\int\limits_{-\pi}^{\pi}\frac{d\varphi}{2\pi}\sin\left(z\sin\frac{\varphi}{2}\right)\sin\frac{\varphi}{2}\cos n\varphi = -J_{2n}'(z)\,.$$

Therefore,

$$P_n = n\frac{\omega e^2}{R}\left[2\beta^2 J_{2n}'(2n\beta) - (1-\beta^2)\int\limits_{0}^{2n\beta}dx\,J_{2n}(x)\right]. \tag{26}$$

The most striking thing about this result is the absence of any marked dependence on energy, at least for small n, while the total power contains the very large factor $(E/mc^2)^4$. The conclusion is irresistible that an enormous number of harmonics must contribute to the total radiation for $E/mc^2 \gg 1$. To verify this contention we shall examine P_n with $n \gg 1$, employing the following approximate Bessel function representations, valid for large order,

$$\begin{aligned} J_{2n}(2n\beta) &= \frac{(1-\beta^2)^{1/2}}{\pi\sqrt{3}}K_{1/3}\left(\frac{2n}{3}(1-\beta^2)^{3/2}\right) \\ &= \frac{1}{\pi\sqrt{3}}\left(\frac{3}{2n}\right)^{1/3}\xi^{1/3}K_{1/3}(\xi), \qquad \xi = \frac{2n}{3}\left(\frac{mc^2}{E}\right)^3 \end{aligned} \tag{27}$$

and

$$J_{2n}'(2n\beta) = \frac{1}{\pi\sqrt{3}}\left(\frac{3}{2n}\right)^{2/3}\xi^{2/3}K_{2/3}(\xi)\,. \tag{28}$$

Hence[6]

$$P_n = \frac{\omega e^2}{R}\left(\frac{2n}{3}\right)^{1/3}\frac{\sqrt{3}}{2\pi}\xi^{2/3}\left[2K_{2/3}(\xi) - \int\limits_{\xi}^{\infty}dx\,K_{1/3}(x)\right], \qquad n \gg 1\,. \tag{29}$$

In virtue of the properties of the Bessel functions of imaginary argument, the behavior of P_n is radically different depending upon whether ξ is large or small compared to unity. Hence a critical harmonic number is

$$n_0 = \left(\frac{E}{mc^2}\right)^3$$

[6] By using basic properties of the K-functions it is straightforward to show that the square bracket in Eq. (29) is $[\cdots] = \int\limits_{\xi}^{\infty} dx\,K_{5/3}(x)$, which yields the more familiar expression for P_n.

which is $\sim 10^7$ for $E = 10^8$ eV. For $n \ll n_0$, $\xi \ll 1$,

$$P_n = \frac{\omega e^2}{R}\left(\frac{n}{3}\right)^{1/3}\frac{\sqrt{3}}{\pi}\,\Gamma\left(\tfrac{2}{3}\right), \qquad 1 \ll n \ll n_0\,. \tag{30}$$

Although supposedly valid only for n large compared with unity, this result is in error by only 15% for $n = 1$, and for $n = 5$, the error has decreased to 5%. When $n \gg n_0$, $\xi \gg 1$,

$$P_n = \frac{\omega e^2}{R}\frac{E}{mc^2}\frac{\sqrt{3}}{2\pi}\sqrt{\frac{\pi\xi}{2}}\,e^{-\xi}, \qquad n \gg n_0\,.$$

Thus the energy radiated in a given harmonic steadily increases with the order of the harmonic until $n \sim n_0$, after which there is a rapid decrease. Since the variation with n in the important region, $n < n_0$, is as $n^{1/3}$, the total power in all harmonics should be proportional to $n_0^{4/3}$, which indeed is $(E/mc^2)^4$. For a more precise check of the formula (29), we calculate

$$\begin{aligned}
P = \sum_{n=1}^{\infty} P_n &\simeq \frac{3}{2}\left(\frac{E}{mc^2}\right)^3 \int_0^\infty d\xi\, P_n \\
&= \frac{\omega e^2}{R}\left(\frac{E}{mc^2}\right)^4 \frac{3\sqrt{3}}{4\pi}\int_0^\infty d\xi\,\xi\left[2K_{2/3}(\xi) - \int_\xi^\infty dx\,K_{1/3}(x)\right] \\
&= \frac{\omega e^2}{R}\left(\frac{E}{mc^2}\right)^4 \frac{3\sqrt{3}}{4\pi}\left[2\int_0^\infty d\xi\,\xi K_{2/3}(\xi) - \frac{1}{2}\int_0^\infty d\xi\,\xi^2 K_{1/3}(\xi)\right] \\
&= \frac{2}{3}\frac{\omega e^2}{R}\left(\frac{E}{mc^2}\right)^4
\end{aligned}$$

with the aid of the integral

$$\int_0^\infty dt\,K_\nu(t)t^{\mu-1} = 2^{\mu-2}\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\mu+\nu}{2}\right).$$

We, therefore, see that the energy is spread over n_0 harmonics with most of the energy appearing in the higher harmonics. The fraction of the power that is radiated in the first harmonic is $\sim (mc^2/E)^4$ which is approximately 10^{-9} for $E = 10^8$ eV. Thus the spectral region in which the energy is predominant is not the fundamental $\lambda = 2\pi R$ (~ 3 m for $R = 0.5$ m), but rather $\lambda \sim 2\pi R/n_0 = 2\pi R(mc^2/E)^3$, which is approximately 4×10^{-5} cm for $R = 0.5$ m and $E = 10^8$ eV. Hence the betatron is a source of visible radiation, rather than ultra-high frequency radio waves.

A knowledge of the angular distribution of the emitted energy is principally of interest in connection with the experimental detection of the radiation. We shall investigate it by the customary Poynting vector procedure. At a distance from the circular orbit large in comparison with the radius R, and, *a fortiori*, with all wavelengths generated by the electron, the expression for the n-th Fourier amplitude of the vector potential, Eq. (19b), can be replaced by

$$\mathbf{A}_n(\mathbf{r}) = \frac{e^{in\omega r/c}}{rc}\int d^3\mathbf{r}'\mathbf{J}_n(\mathbf{r}')\,e^{-in\omega\hat{\mathbf{n}}\cdot\mathbf{r}'/c}\,, \qquad r \gg R$$

where $\hat{\mathbf{n}}$ is a unit vector directed toward the point of observation. The scalar potential is most conveniently expressed in terms of the vector potential by means of the Lorentz condition, which makes the following statement about the Fourier amplitudes:

$$\nabla\cdot\mathbf{A}_n = i\frac{n\omega}{c}\phi_n\,.$$

Under the conditions contemplated, the gradient operator can be replaced by $i(n\omega/c)\hat{\mathbf{n}}$, for it effectively acts only on the rapidly oscillating $e^{in\omega r/c}$. Hence $\phi_n = \hat{\mathbf{n}}\cdot\mathbf{A}_n$ and, in consequence, the electric field Fourier amplitude is (*cf.* Eq. (20))

$$\mathbf{E}_n = i\frac{n\omega}{c}(\mathbf{A}_n - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{A}_n)) = -i\frac{n\omega}{c}\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{A}_n)$$

and the magnetic field amplitude is, similarly,

$$\mathbf{H}_n = \nabla\times\mathbf{A}_n = i\frac{n\omega}{c}\hat{\mathbf{n}}\times\mathbf{A}_n$$

which together express the usual relations between the electric and magnetic fields in the wave zone. The average flux is

$$\mathbf{S} = \frac{c}{4\pi}\,\overline{\mathbf{E}\times\mathbf{H}} = \frac{c}{4\pi}\sum_{n=-\infty}^{\infty}\mathbf{E}_n^*\times\mathbf{H}_n = \frac{c}{2\pi}\mathrm{Re}\left\{\sum_{n=1}^{\infty}\mathbf{E}_n^*\times\mathbf{H}_n\right\}$$

and hence the energy flux associated with the n-th harmonic is

$$\mathbf{S}_n = \frac{c}{2\pi}\mathrm{Re}\{\mathbf{E}_n^*\times\mathbf{H}_n\} = \hat{\mathbf{n}}\,\frac{c}{2\pi}\,|\mathbf{E}_n|^2 = \hat{\mathbf{n}}\frac{n^2\omega^2}{2\pi c}\left[|\mathbf{A}_n|^2 - |\hat{\mathbf{n}}\cdot\mathbf{A}_n|^2\right]$$

The integral expression for $\mathbf{A}_n$ can be immediately simplified to

$$\mathbf{A}_n(\mathbf{r}) = \frac{e^{in\omega r/c}}{r}e\beta\int_{-\pi}^{\pi}\frac{d\varphi'}{2\pi}\,e^{-in\beta\sin\theta\cos(\varphi-\varphi')}\,\mathbf{e}_{\varphi'}\,e^{in(\varphi'-\varphi_0)}$$

on inserting (18) and integrating with respect to ρ and z. Here θ and φ are the polar angles of the observation point, $\theta = 0$ corresponding to the positive z axis. It must be remembered, in performing the φ' integration, that $\mathbf{e}_{\varphi'}$ is a variable vector, which, however, can be resolved into the constant radial and angular unit vectors associated with the point of observation,

$$\mathbf{e}_{\varphi'} = \mathbf{e}_\varphi\cos(\varphi-\varphi') + \mathbf{e}_\rho\sin(\varphi-\varphi')$$

Hence,

$$\mathbf{A}_n(\mathbf{r}) = \frac{e^{in\omega r/c}}{r}e\beta e^{in(\varphi-\varphi_0)}\left[\mathbf{e}_\varphi\int_{-\pi}^{\pi}\frac{d\varphi'}{2\pi}\,e^{-in\beta\sin\theta\cos\varphi'}\cos\varphi' e^{in\varphi'} - \mathbf{e}_\rho\int_{-\pi}^{\pi}\frac{d\varphi'}{2\pi}\,e^{-in\beta\sin\theta\cos\varphi'}\sin\varphi' e^{in\varphi'}\right]$$

where we have also replaced $\varphi'-\varphi$ by φ' as the variable of integration. From the well-known integral

$$\int_{-\pi}^{\pi}\frac{d\varphi'}{2\pi}\,e^{-ix\cos\varphi'}e^{in\varphi'} = i^{-n}J_n(x)$$

we derive

$$\int_{-\pi}^{\pi}\frac{d\varphi'}{2\pi}\,e^{-ix\cos\varphi'}\cos\varphi' e^{in\varphi'} = i^{1-n}J_n'(x)$$

$$\int_{-\pi}^{\pi}\frac{d\varphi'}{2\pi}\,e^{-ix\cos\varphi'}\sin\varphi' e^{in\varphi'} = -i^{-n}\frac{n}{x}J_n(x)$$

by differentiation with respect to x, and integration by parts, respectively. Therefore,

$$\mathbf{A}_n(\mathbf{r}) = e\beta\,\frac{e^{in\omega r/c}}{r} e^{in(\varphi-\varphi_0-\pi/2)}\left[i\,\mathbf{e}_\varphi J_n'(n\beta\sin\theta) + \mathbf{e}_\rho \frac{J_n(n\beta\sin\theta)}{\beta\sin\theta}\right]$$

and

$$\begin{aligned}|\mathbf{A}_n|^2 &= \frac{e^2\beta^2}{r^2}\left[J_n'^2(n\beta\sin\theta) + \frac{J_n^2(n\beta\sin\theta)}{\beta^2\sin^2\theta}\right] \\ |\hat{\mathbf{n}}\cdot\mathbf{A}_n|^2 &= \frac{e^2\beta^2}{r^2}\sin^2\theta\left[\frac{J_n^2(n\beta\sin\theta)}{\beta^2\sin^2\theta}\right].\end{aligned}$$

Thus, finally, the power radiated in the n-th harmonic into a unit solid angle about the direction $\hat{\mathbf{n}}$ is

$$P_n(\hat{\mathbf{n}}) = r^2\,|\mathbf{S}_n| = \frac{\omega e^2}{2\pi R}\beta^3 n^2\left[J_n'^2(n\beta\sin\theta) + \cos^2\theta\,\frac{J_n^2(n\beta\sin\theta)}{\beta^2\sin^2\theta}\right] \tag{31}$$

which naturally depends only upon the angle θ.

As a check upon this result we shall verify that the total power radiated in the n-th harmonic into a unit solid angle about the direction $\hat{\mathbf{n}}$ is in agreement with Eq. (26). To this end we multiply (31) by the element of solid angle $2\pi\sin\theta d\theta$ and integrate with respect to θ from 0 to π:

$$P_n = \frac{\omega e^2}{R}\beta^3 n^2\int_0^\pi d\theta\,\sin\theta\left[J_n'^2(n\beta\sin\theta) + \cos^2\theta\,\frac{J_n^2(n\beta\sin\theta)}{\beta^2\sin^2\theta}\right]. \tag{32}$$

To evaluate these integrals we employ the formula

$$\int_0^{2\pi}\frac{d\varphi}{2\pi}\,J_{2n}(2x\cos\varphi) = J_n^2(x)\,.$$

Thus

$$\begin{aligned}\int_0^\pi d\theta\sin\theta\,J_n^2(n\beta\sin\theta) &= \int_0^\pi d\theta\sin\theta\int_0^{2\pi}\frac{d\varphi}{2\pi}\,J_{2n}(2n\beta\sin\theta\cos\varphi) \\ &= \int_0^\pi d\theta\sin\theta\int_0^{2\pi}\frac{d\varphi}{2\pi}\,J_{2n}(2n\beta\cos\theta) = \frac{1}{n\beta}\int_0^{2n\beta}dx\,J_{2n}(x)\end{aligned}$$

where the third integral is derived from the second by regarding it as an integral extended over the surface of a unit sphere. The replacement of $\sin\theta\cos\phi$, the x-coordinate of a point on the sphere, by $\cos\theta$, the z-coordinate, amounts to a rotation of the coordinate system, which does not affect the value of the integral. By combining the Bessel function recurrence relations

$$\begin{aligned}2J_n'(x) &= J_{n-1}(x) - J_{n+1}(x) \\ \frac{2n}{x}J_n(x) &= J_{n-1}(x) + J_{n+1}(x)\end{aligned}$$

into

$$J_n'^2(x) + \left(\frac{n}{x}\right)^2 J_n^2(x) = \frac{1}{2}\left(J_{n-1}^2(x) + J_{n+1}^2(x)\right)$$

we obtain, in a similar way,

$$\int_0^\pi d\theta\, \sin\theta \left[J_n'^2(n\beta\sin\theta) + \frac{J_n^2(n\beta\sin\theta)}{\beta^2\sin^2\theta} \right]$$

$$= \frac{1}{2}\int_0^\pi d\theta\, \sin\theta \int_0^{2\pi} \frac{d\varphi}{2\pi}\, \left[J_{2n+2}(2n\beta\sin\theta\cos\varphi) + J_{2n-2}(2n\beta\sin\theta\cos\varphi)\right]$$

$$= \frac{1}{2n\beta}\int_0^{2n\beta} dx\, \left[J_{2n+2}(x) + J_{2n-2}(x)\right] = \frac{1}{n\beta}\int_0^{2n\beta} dx\, J_{2n}(x) + \frac{2}{n\beta} J_{2n}'(2n\beta)\,.$$

The last transformation requires the use of

$$\tfrac{1}{2}\left(J_{2n-2}(x) + J_{2n+2}(x)\right) = J_{2n}(x) + 2J_{2n}''(x)\,.$$

Combining the two integrals, we reach the desired result.

The angular distribution of the high harmonics is of principal interest, for we have seen that little energy is radiated in the longer wave lengths. The general character of the radiation pattern is easily seen, for Bessel functions of high order are very small if the argument is appreciably less than the order, and therefore the radiation intensity is negligible unless $\sin\theta$ is close to unity. Hence the radiation is closely confined to the plane of the orbit. For a more precise analysis, we employ the approximate Bessel function representation already described (Eqs. (27) and (28)). Introducing the angle[7] $\psi = \pi/2 - \theta$ between the point of observation and the plane of the orbit, which we suppose to be small, we write the power radiated into a unit angular range about the angle ψ, in the n-th harmonic, as

$$P_n(\psi) = \frac{\omega e^2}{R}\frac{3}{\pi^2}\left(\frac{n}{3}\right)^{2/3}\left[\xi^{4/3}K_{2/3}^2(\xi) + \psi^2\left(\frac{n}{3}\right)^{2/3}\xi^{2/3}K_{1/3}^2(\xi)\right]$$

where

$$\xi = \frac{n}{3}\left(1 - \beta^2\sin^2\theta\right)^{3/2} \simeq \frac{n}{3}\left(\psi^2 + \left(\frac{mc^2}{E}\right)^2\right)^{3/2}$$

In consequence of the properties of the cylinder functions, the radiation intensity decreases rapidly when ξ becomes appreciably greater than unity. Hence, for the modes of importance, $n < (E/mc^2)^3$, the angular range within which the energy is sensibly confined is of the order $n^{-1/3}$. Within this angular range the intensity per unit angle is essentially independent of ψ and varies with n as $n^{2/3}$, which is consistent with the $n^{1/3}$ variation of the total power in a given harmonic. In virtue of the concentration of the radiation at the higher harmonics, $n \sim n_0$, it is clear that the mean angular range for the total radiation will be $\sim n_0^{-1/3} = mc^2/E$, which is approximately 0.3° for $E = 10^8$ eV.

Although this result is quite informative, it does not give a complete picture of the radiation angular distribution, for it is concerned only with the average properties, giving no information about the instantaneous radiation pattern of an electron pursuing the circular trajectory. We shall now show that the electron radiates within a cone of angle $\sim mc^2/E$ drawn about the instantaneous direction of motion. Thus, an observer stationed in the plane of the orbit will only detect radiation emitted from that small part of the path within which the velocity of the electron is directed toward the point of observation. To prove this contention, we appeal to the well known retarded potentials of a point charge moving in an arbitrary manner,

$$\phi(\mathbf{r}, t) = \frac{e}{|\mathbf{r} - \mathbf{r}_e(\tau)| - \dfrac{\mathbf{v}(\tau)}{c}\cdot(\mathbf{r} - \mathbf{r}_e(\tau))}$$

[7] The manuscript actually says $\psi = \theta - \pi/2$; although the sign difference in the definition of ψ is of no consequence in the context of this paper, I chose to follow the conventional definition.

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}(\tau)}{c}\phi(\mathbf{r}, t)$$

where $\mathbf{r}_e(\tau)$ is the position of the electron at τ, the time of emission of the field which reaches the point $\mathbf{r}$ at the time t, and

$$\tau = t - \frac{|\mathbf{r} - \mathbf{r}_e(\tau)|}{c}.$$

In calculating the fields from these potentials, it must be remembered that τ is an implicit function of $\mathbf{r}$ and t. Thus

$$\begin{aligned} \mathbf{H} = \nabla \times \mathbf{A} &= \phi \nabla \times \frac{\mathbf{v}}{c} - \frac{\mathbf{v}}{c} \times \nabla\phi \\ &= \frac{1}{c}\phi\nabla\tau \times \frac{d\mathbf{v}}{d\tau} - \left(\frac{\partial\phi}{\partial\tau}\right)_{\mathbf{r}} \frac{\mathbf{v}}{c} \times \nabla\tau - \frac{\mathbf{v}}{c} \times (\nabla\phi)_\tau \end{aligned} \tag{33}$$

where the bracket symbols signify that the quantity indicated as a subscript is kept constant during the differentiation. We are concerned only with the radiation field—that part of the field varying inversely with the distance from the emission point of the electron—and therefore the last term of (33) will be discarded. Now

$$\nabla\tau = -\frac{1}{c}\,\frac{\mathbf{r} - \mathbf{r}_e}{|\mathbf{r} - \mathbf{r}_e| - \frac{\mathbf{v}}{c} \cdot (\mathbf{r} - \mathbf{r}_e)}$$

$$\left(\frac{\partial\phi}{\partial\tau}\right)_{\mathbf{r}} = \frac{e}{c}\,\frac{\frac{d\mathbf{v}}{d\tau} \cdot (\mathbf{r} - \mathbf{r}_e)}{\left[|\mathbf{r} - \mathbf{r}_e| - \frac{\mathbf{v}}{c} \cdot (\mathbf{r} - \mathbf{r}_e)\right]^2}$$

and, hence, with the notation

$$|\mathbf{r} - \mathbf{r}_e| = r, \qquad \frac{\mathbf{r} - \mathbf{r}_e}{|\mathbf{r} - \mathbf{r}_e|} = \hat{\mathbf{n}}, \qquad \frac{\mathbf{v}}{c} \cdot \hat{\mathbf{n}} = \beta\cos\theta, \qquad \frac{d\mathbf{v}}{d\tau} = \dot{\mathbf{v}}$$

we find

$$\mathbf{H} = -\frac{e}{rc^2}\,\frac{1}{(1 - \beta\cos\theta)^3}\,\hat{\mathbf{n}} \times \left[\dot{\mathbf{v}} + \hat{\mathbf{n}} \times \left(\frac{\mathbf{v}}{c} \times \dot{\mathbf{v}}\right)\right].$$

The magnitude of the Poynting vector is then

$$S = \frac{c}{4\pi}\mathbf{H}^2 = \frac{e^2}{4\pi r^2 c^3}\,\frac{1}{(1 - \beta\cos\theta)^6}\left[\dot{\mathbf{v}}^2 - (\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})^2 + \left(\frac{\mathbf{v}}{c} \times \dot{\mathbf{v}}\right)^2 - \left(\hat{\mathbf{n}} \cdot \frac{\mathbf{v}}{c} \times \dot{\mathbf{v}}\right)^2 + 2\frac{\mathbf{v} \cdot \dot{\mathbf{v}}}{c}\,\hat{\mathbf{n}} \cdot \dot{\mathbf{v}} - 2\dot{\mathbf{v}}^2\,\hat{\mathbf{n}} \cdot \frac{\mathbf{v}}{c}\right].$$

It might be thought that this is the solution to the problem—but it is not. The quantity S, when multiplied by r^2, gives the energy radiated in a unit solid angle about the direction $\hat{\mathbf{n}}$, in a unit interval of the observer's time t. However, we are interested in the energy emitted in a given direction during a unit interval of radiating time τ. It is necessary, therefore, to correct S by the factor

$$\frac{dt}{d\tau} = 1 - \beta\cos\theta \tag{34}$$

and thus the power radiated into a unit solid angle about the direction $\hat{\mathbf{n}}$ is

$$P(\hat{\mathbf{n}}) = \frac{e^2}{4\pi c^3}\,\frac{1}{(1 - \beta\cos\theta)^5}\left[\dot{\mathbf{v}}^2 - (\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})^2 + \left(\frac{\mathbf{v}}{c} \times \dot{\mathbf{v}}\right)^2 - \left(\hat{\mathbf{n}} \cdot \frac{\mathbf{v}}{c} \times \dot{\mathbf{v}}\right)^2 + 2\frac{\mathbf{v} \cdot \dot{\mathbf{v}}}{c}\,\hat{\mathbf{n}} \cdot \dot{\mathbf{v}} - 2\dot{\mathbf{v}}^2\,\hat{\mathbf{n}} \cdot \frac{\mathbf{v}}{c}\right] \tag{35}$$

To confirm the necessity of including the factor (34), we shall calculate the total power radiated in all directions and verify that it is in agreement with (7). For this purpose, it is sufficient to consider the particular case where the electron is accelerated in the direction of motion ($\mathbf{v}\times\dot{\mathbf{v}}=0$). We must then calculate

$$P = 2\pi\int_0^\pi d\theta\sin\theta P(\hat{\mathbf{n}}) = \frac{e^2}{2c^3}\dot{\mathbf{v}}^2\int_0^\pi d\theta\sin\theta\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

The result of the integration is

$$P = \frac{2}{3}\frac{e^2}{c^3}\frac{\dot{\mathbf{v}}^2}{(1-\beta^2)^3}$$

which is easily shown to agree with Eq. (7). If, however, the factor $1-\beta\cos\theta$ had not been included, which is the course followed by Sommerfeld in treating this problem [3], the result would be

$$P = \frac{2}{3}\frac{e^2}{c^3}\dot{\mathbf{v}}^2\frac{1+\frac{1}{5}\beta^2}{(1-\beta^2)^4}$$

To apply Eq. (35) most conveniently to the problem of an electron pursuing a circular trajectory with constant speed, choose the direction of the velocity at a particular instant as the z axis of a coordinate system, and the direction of the negative acceleration as the x axis. The xz plane coincides with the plane of the orbit. The power radiated per unit solid angle in a direction specified by the polar angles θ and φ, relative to this coordinate system, is then

$$P(\theta,\varphi) = \frac{e^2}{4\pi c^3}\dot{\mathbf{v}}^2\left[\frac{1}{(1-\beta\cos\theta)^3} - (1-\beta^2)\frac{\sin^2\theta\cos^2\varphi}{(1-\beta\cos\theta)^5}\right]$$

which evidently makes the direction of motion a strongly preferred direction of emission. As a simple measure of this asymmetry, note that the ratio of the intensity in the forward direction to that at right angles to the direction of motion is approximately $(1-\beta)^{-3}\simeq 8(E/mc^2)^6$, which is 5×10^{14} for $E=10^8$ eV. As a more useful measure of the concentration of energy, we shall calculate the mean value of $\sin^2\theta$, defined as

$$\overline{\sin^2\theta} = \frac{1}{P}\int \sin^2\theta P(\theta,\varphi)\sin\theta d\theta d\varphi\,.$$

An elementary calculation yields, in the limit of high energies:

$$\overline{\sin^2\theta}\simeq\overline{\theta^2} = \left(\frac{mc^2}{E}\right)^2$$

and therefore the mean angle between the direction of emission and the direction of the electron's motion is mc^2/E.

The angular distribution properties of the radiation suggest a simple physical explanation for the very large number of harmonics generated by the electron. Since the mean angle between the direction of the motion and that of the radiation is $\Delta\theta = mc^2/E$, the time interval during which radiation is emitted toward the observer is

$$\Delta\tau\sim\frac{\Delta\theta}{\omega}$$

being the time required for the direction of motion to move through the angle $\Delta\theta$. As the electron moves around the circular path a series of such pulses, separated by the time interval $2\pi/\omega$, will be directed toward the point of observation. Hence, one might argue, the radiation will consist of a sequence of angular frequencies, $n\omega$, the effective maximum frequency being determined by the pulse duration

$$\omega_{\max} = n_0\omega\sim\frac{1}{\Delta\tau}$$

or

$$n_0\sim\frac{1}{\Delta\theta} = \frac{E}{mc^2}$$

However, this argument is fallacious at one point: the time duration of the pulse received by the observer, Δt, is not equal to $\Delta\tau$, the time interval during which the radiation is emitted. This consequence of the Doppler effect has already been mentioned and is described by Eq. (34). The latter equation is simplified approximately by remarking that we are concerned with small angles ($\cos\theta \simeq 1-\theta^2/2$), and speeds close to that of light ($\beta \simeq 1-(mc^2/E)^2/2$) whence

$$\Delta t = \frac{\Delta\tau}{2}\left[\left(\frac{mc^2}{E}\right)^2 + \theta^2\right] \sim \Delta\tau\left(\frac{mc^2}{E}\right)^2 = \frac{1}{\omega}\left(\frac{mc^2}{E}\right)^3 .$$

Hence, the equation to determine the maximum harmonics now correctly reads

$$\omega_{\max} = n_0\omega \sim \frac{1}{\Delta t} = \omega\left(\frac{E}{mc^2}\right)^3$$

or $n_0 \sim (E/mc^2)^3$, in agreement with our previous considerations.

With this stock of information concerning the radiation by a single electron, we are prepared to discuss the modifications introduced by the simultaneous presence of many electrons in the betatron. Consider N electrons traversing the circular orbit, the angular position of the k-th electron at time t being $\varphi_k + \omega t$. When the mutual action of the electrons is negligible, φ_k is constant and specifies the angular position of the electron at the arbitrary time $t = 0$. The charge density of the N electrons is obtained by the addition of the individual densities and is therefore of the form (17), with

$$\rho_n(\mathbf{r}) = e\frac{\delta(\rho - R)}{R}\delta(z)\frac{e^{in\varphi}}{2\pi}\sum_{k=1}^{N} e^{-in\varphi_k}$$

which only differs from the corresponding one electron expression in the replacement of the phase factor $e^{-in\varphi_0}$ by the sum

$$\sum_{k=1}^{N} e^{-in\varphi_k}$$

A similar remark applies to the current density Fourier amplitude. It should then be clear that the rate of radiation into the n-th harmonic by N electrons, $P_n^{(N)}$, is related to P_n, that of one electron, by

$$P_n^{(N)} = P_n\left|\sum_{k=1}^{N} e^{-in\varphi_k}\right|^2$$

It is evident that this expression does imply the possibility of a substantial reduction of the radiation by destructive interference, if the electrons are properly arranged on the path. As an extreme example, suppose the electrons to be uniformly spaced on the circular trajectory:

$$\varphi_k = \frac{2\pi}{N}(k-1), \qquad k = 1, \cdots, N .$$

Then

$$\sum_{k=1}^{N} e^{-in\varphi_k} = \frac{1-e^{2\pi i n}}{1-e^{2\pi i n/N}} = \begin{cases} 0 & \text{if } 1 \le n \le N-1,\ N+1 \le n \le 2N-1, \cdots \\ N & \text{if } n = N,\ 2N, \cdots \end{cases}$$

and all harmonics up to the N-th are completely suppressed. Therefore if N is appreciably greater than the critical harmonic, $n_0 = (E/mc^2)^3$, the radiation is practically eliminated. It is interesting to note, however, that if $N \sim n_0$, all harmonics will be suppressed save the N-th, and the power radiated into it will be N^2 times that generated by a single electron. Hence

$$P^{(N)} \simeq P_N^{(N)} \sim N^2 \frac{\omega e^2}{R} n_0^{1/3} \sim N \frac{\omega e^2}{R} \left(\frac{E}{mc^2}\right)^4$$

which is essentially identical with the power radiated by N independent electrons. For $E = 10^8$ eV and $R = 0.5$ m, this situation corresponds to a circulating current of 100 μA. If $E = 10^9$ eV, $R = 5$ m, the circulating current must considerably exceed 10 mA if the radiation is to be reduced, even in this most extreme circumstance.

In the actual situation, however, we must certainly regard the electrons as uncorrelated in position and randomly distributed around the circular path. Therefore, to obtain the average radiation in the n-th harmonic, we must calculate the mean value of

$$C_n = \left| \sum_{k=1}^{N} e^{-in\varphi_k} \right|^2 = N + \sum_{j \neq k} \cos n(\varphi_j - \varphi_k) \tag{36}$$

averaged over all values of the phases φ_k. Clearly $\overline{C}_n = N$, and the average radiation in any harmonics is just N times that produced by a single electron. Hence, although the average field intensity is zero and no coherent radiation (proportional to N^2) exists, as the elementary argument mentioned in the introduction properly predicts, the incoherent radiation effects of the individual electrons remain. The latter may then be considered a fluctuation phenomenon, akin to the shot effect and thermal noise. It is important to notice that the radiation is subject to large fluctuations, for the mean value of C_n^2 is

$$\overline{C_n^2} = N^2 + N(N-1)$$

and therefore the root mean square deviation of C_n is

$$\sqrt{\overline{C_n^2} - \overline{C}_n^2} = \sqrt{N(N-1)} \simeq \overline{C}_n, \qquad N \gg 1\,. \tag{37}$$

For large N, the probability distribution of all the C_n's is given by [4]

$$W(C)dC = e^{-C/N}\,\frac{dC}{N} \tag{38}$$

and, hence, the most probable value of the radiation is zero.

The situation is quite different if the electrons are not uniformly distributed around the circle, for in this case coherent radiation exists in addition to the incoherent radiation of the individual electrons. Such a state of affairs will exist in any of the resonance acceleration schemes that have been proposed for the production of high energy electrons [5], since in such a device only electrons having the proper phase relations with an alternating electric field will be accelerated, and these will occupy only a small portion of the orbit. It is not our intention to elaborate on any of these methods; we are only concerned with the radiation to be anticipated from a pulse of electrons traversing a circular path.

We shall suppose that the electrons are uniformly distributed over an angular range α. To calculate the average radiation emitted by N electrons in the n-th harmonic, it is necessary to evaluate[8] (36) averaged over all angular positions of each electron within the interval $-\alpha/2$ to $\alpha/2$. In view of the independence of the electrons,

$$\overline{\left| \sum_{k=1}^{N} e^{-in\varphi_k} \right|^2} = N + N(N-1) \left[\frac{1}{\alpha} \int_{-\alpha/2}^{\alpha/2} d\varphi \cos n\varphi \right]^2 = N + N(N-1) \left[\frac{\sin n\alpha/2}{n\alpha/2} \right]^2$$

[8]The original manuscript duplicates Eq. (36) at this point.

which indicates explicitly the incoherent and coherent parts of the radiation. The total coherent radiation by N electrons is

$$P_{\text{coh.}}^{(N)} = N^2 \sum_{n=1}^{\infty} \left[\frac{\sin n\alpha/2}{n\alpha/2} \right]^2 P_n, \qquad N \gg 1 \tag{39}$$

which we shall evaluate under the conditions $1 \gg \alpha \gg 1/n_0$, that is, the spatial length of the pulse is small compared to the radius of the orbit, but large compared to the shortest wavelength effectively emitted by a single electron. The interference factor $(\sin x/x)^2$, $x = n\alpha/2$, is unity for $x \ll 1$, but decreases fairly rapidly for larger values of x. The effective upper limit of the summation in (39) is $\sim n_0$, but since $n_0\alpha \gg 1$, little error is produced by replacing P_n with an expression valid for $n < n_0$ and extending the summation to infinity. Further, since Eq. (30) is not seriously in error even for $n = 1$, we can write

$$P_{\text{coh.}}^{(N)} = N^2 \frac{\omega e^2}{R} \frac{3^{1/6}}{\pi} \Gamma(\tfrac{2}{3}) \sum_{n=1}^{\infty} \left[\frac{\sin n\alpha/2}{n\alpha/2} \right]^2 n^{1/3}. \tag{40}$$

An order of magnitude for (40) is easily obtained, for the n-th term of the summation is practically $n^{1/3}$ for $n < 1/\alpha$ and rapidly decreases for $n > 1/\alpha$. Hence the summation is roughly $(1/\alpha)^{4/3}$, and the coherent radiation is approximately

$$P_{\text{coh.}}^{(N)} \sim N^2 \frac{\omega e^2}{R} \left(\frac{1}{\alpha} \right)^{4/3} \tag{41}$$

which, it should be noted, is independent of energy, except to the extent that phase focusing produces a decrease of α with increasing energy. Before further discussion, we shall evaluate the summation in (40) more precisely and thus obtain the numerical factors which are omitted in (41). It can be verified, by contour integration, that

$$\begin{aligned}
\sum_{n=1}^{\infty} \left[\frac{\sin n\alpha/2}{n\alpha/2} \right]^2 n^{1/3} &= \frac{2}{\alpha^2} \int_0^{\infty} dt \, \frac{\sinh(\alpha t/2) \, \sinh((\pi - \alpha/2)t)}{\sinh \pi t} \, t^{-5/3} \\
&\simeq \frac{2}{\alpha^2} \int_0^{\infty} dt \sinh(\alpha t/2) \, e^{-\alpha t/2} \, t^{-5/3}, \qquad \alpha \ll 1 \\
&= \frac{1}{\alpha^2} \int_0^{\infty} dt \left(1 - e^{-\alpha t}\right) t^{-5/3}, \\
&= \tfrac{3}{2} \Gamma\left(\tfrac{1}{3}\right) \alpha^{-4/3}.
\end{aligned}$$

Hence,[9]

$$P_{\text{coh.}}^{(N)} = N^2 \frac{\omega e^2}{R} \left(\frac{\sqrt{3}}{\alpha} \right)^{4/3} \tag{42}$$

which is to be compared with

$$P_{\text{incoh.}}^{(N)} = N \frac{2}{3} \frac{\omega e^2}{R} \left(\frac{E}{mc^2} \right)^4$$

At low energies, the coherent radiation is much more important than the incoherent radiation. The two effects become equal at an energy determined by

$$\frac{E}{mc^2} = \left(\frac{3N}{2} \right)^{1/4} \left(\frac{\sqrt{3}}{\alpha} \right)^{1/3}.$$

[9] A shortcut method to obtain this result, which is justified and used elsewhere in this paper, is to regard n as a continuous variable and to replace the summation in Eq. (40) by an integration from 0 to ∞.

As a numerical example, let $\alpha = 10^{-2}$ which, for a radius of 5 m, represents a pulse length of 5 cm, and $N = 10^9$, corresponding to an average circulating current ~ 1 mA. Then $E/mc^2 = 10^3$, or $E = 5 \times 10^8$ eV.

The power emitted in coherent radiation can be described by a radiation resistance

$$\mathcal{R} = 120\pi^2 \left(\frac{\sqrt{3}}{\alpha}\right)^{4/3} \quad \Omega$$

which, multiplied by the square of the average circulating current in amperes, gives the radiated power in watts. For the numbers considered above, $\mathcal{R} = 10^6\ \Omega$ and the coherent power is of the order of one watt. The voltage required to drive the current I through the resistance $\mathcal{R}$, $V = \mathcal{R}I$, is equal to the coherent energy loss of an electron, per revolution, measured in eV. Thus, under the stated conditions, $(\delta E)_{\text{coh.}} \sim 10^3$ eV. It is important to note that, since the radiation counter-voltage $\mathcal{R}I$ is independent of the electron energy (to the extent that α does not change with energy), it effectively reduces the voltage of the accelerating electric field by a constant fraction. Hence, coherent radiation represents a loss in efficiency, not an insurmountable barrier to the attainment of high energies. Furthermore, since the coherent radiation is emitted at long wavelengths, it can be influenced—and reduced—by external means. We have seen that the maximum effective harmonic is $n \sim 1/\alpha$, which corresponds to a wavelength of the order of the pulse length. If, for example, $R = 5$ m and $\alpha = 10^{-2}$, the coherent spectrum contains some hundred harmonics, and extends from the fundamental wavelength $\lambda \sim 30$ m down to wavelengths of a few centimeters. This ultra high frequency and microwave radiation will be strongly affected by the presence of metal close to the orbit of the electrons.

As a simple example, we shall consider the radiation by the electrons in the presence of two plane sheets of metal placed parallel to the plane of the orbit. The distance between the sheets will be denoted by a and it will be supposed that the orbit plane is equidistant from each metallic sheet. Such a parallel plate metallic system acts like a waveguide with the fundamental property that radiation with a wavelength greater than $2a$ cannot be propagated, but rather is exponentially attenuated with increasing distance from the source. The wavelength associated with the n-th harmonic is $\lambda = 2\pi R/n\beta$, and therefore all harmonics with $n < \pi R/\beta a$ will be completely suppressed. Hence, if a is less than the pulse length, the coherent radiation will be largely eliminated.

To begin the mathematical analysis of the situation we note that, in computing the rate at which an electron transfers energy to the field, the relevant components of the electric field are those parallel to the plane of the orbit, and these must vanish on the two metallic surfaces, which are supposed for the moment to be of infinite conductivity. Hence, the quantity that determines the potential Fourier amplitudes of a given charge distribution,

$$\mathcal{G}(\mathbf{r}, \mathbf{r}') = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|}, \qquad k = \frac{n\omega}{c}$$

the Green's function of free space, which satisfies the inhomogeneous wave equation

$$(\nabla^2 + k^2)\mathcal{G}(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}')$$

must be replaced by a solution of this equation subject to the boundary conditions

$$\mathcal{G}(\mathbf{r}, \mathbf{r}') = 0, \quad z = -a/2,\ a/2\,.$$

With no loss in generality, we may temporarily suppose the point $\mathbf{r}'$ to be located on the z axis; the Green's function is then axially symmetric and satisfies the polar form of the inhomogeneous wave equation:

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + k^2 + \frac{\partial^2}{\partial z^2}\right)\mathcal{G}(\mathbf{r}, \mathbf{r}') = -2\frac{\delta(\rho)}{\rho}\delta(z - z')\,. \tag{43}$$

We shall solve this equation by representing the z dependence of the Green's function with the aid of the complete set of orthogonal functions appropriate to the boundary conditions

$$\sin\frac{j\pi}{a}\left(z+\frac{a}{2}\right), \qquad j=1,\ 2,\ \cdots \tag{44}$$

Thus we write

$$\mathcal{G}(\mathbf{r},\mathbf{r}') = \frac{2}{a}\sum_{j=1}^{\infty}\sin\frac{j\pi}{a}\left(z+\frac{a}{2}\right)\sin\frac{j\pi}{a}\left(z'+\frac{a}{2}\right)f_j(\rho)$$

anticipating that the function $f_j(\rho)$ is truly a function only of ρ. Substituting this expansion into Eq. (43) and employing the orthogonality properties of the functions (44) to isolate the equation satisfied by a particular $f_j(\rho)$, we find

$$\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}+k^2-\left(\frac{j\pi}{a}\right)^2\right]f_j(\rho) = -2\frac{\delta(\rho)}{\rho} \tag{45}$$

which for $\rho > 0$ is evidently satisfied by a cylinder function of order zero and argument $\sqrt{k^2-(j\pi/a)^2}\,\rho$. The required function is

$$f_j(\rho) = CH_0^{(1)}\left(\sqrt{k^2-(j\pi/a)^2}\,\rho\right)$$

for this choice correctly meets the boundary requirement that only waves moving away from the source shall occur (the radiation condition). The constant C is fixed by the inhomogeneous term which represents the source strength. Multiply Eq. (45) by ρ and integrate from 0 to some arbitrary radius ρ; in consequence of the delta function property,

$$\rho\frac{d}{d\rho}f_j(\rho) + \left[k^2-\left(\frac{j\pi}{a}\right)^2\right]\int_0^{\rho} d\rho'\rho' f_j(\rho') = -2$$

which must be valid for any ρ. In the limit as $\rho\to 0$, the integral becomes negligibly small, and the first term approaches $2iC/\pi$. Hence $C = i\pi$ and the Green's function is, finally,

$$\mathcal{G}(\mathbf{r},\mathbf{r}') = \frac{2\pi i}{a}\sum_{j=1}^{\infty}\sin\frac{j\pi}{a}\left(z+\frac{a}{2}\right)\sin\frac{j\pi}{a}\left(z'+\frac{a}{2}\right)H_0^{(1)}\left(\sqrt{k^2-(j\pi/a)^2}\,\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')}\right),$$

abandoning the temporary restriction that $\rho' = 0$. This result can also be rewritten with the aid of the cylinder function addition theorem,

$$\begin{aligned}\mathcal{G}(\mathbf{r},\mathbf{r}') = \frac{2\pi i}{a}\sum_{j=1}^{\infty}\sum_{m=-\infty}^{\infty}&\sin\frac{j\pi}{a}\left(z+\frac{a}{2}\right)\sin\frac{j\pi}{a}\left(z'+\frac{a}{2}\right)\\ &\times H_m^{(1)}\left(\sqrt{k^2-(j\pi/a)^2}\,\rho_>\right)J_m\left(\sqrt{k^2-(j\pi/a)^2}\,\rho_<\right)e^{im(\varphi-\varphi')}\end{aligned}$$

where $\rho_<$ and $\rho_>$ are respectively, the smaller and larger of the two radii ρ and ρ'. The latter formula can be obtained directly by the same methods.

To calculate the average power radiated into the n-th harmonic by a single electron moving within the parallel plates, we return to Eq. (21) and replace the free space Green's function by the one just determined. The ensuing steps to Eq. (22) are unchanged save that

$$\frac{e^{2in\beta|\sin(\varphi/2)|}}{\left|\sin\frac{\varphi}{2}\right|}$$

must be replaced by

$$
\begin{aligned}
&i\frac{4\pi R}{a}\sum_{j=1,3,\cdots}^{\infty} H_0^{(1)}\left(2\sqrt{(n\beta)^2-(j\pi R/a)^2}\left|\sin\frac{\varphi}{2}\right|\right) \\
&= i\frac{4\pi R}{a}\sum_{j=1,3,\cdots}^{\infty}\sum_{m=-\infty}^{\infty} H_m^{(1)}\left(\sqrt{(n\beta)^2-(j\pi R/a)^2}\right) J_m\left(\sqrt{(n\beta)^2-(j\pi R/a)^2}\right)e^{im\varphi}.
\end{aligned}
$$

Performing the integration with respect to φ, using the latter form of Green's function, we find

$$
P_n = n\frac{\omega e^2}{R}\frac{4\pi R}{a}\mathrm{Re}\left\{\sum_{j=1,3,\cdots}^{\infty}\left[-H_n^{(1)}J_n+\frac{\beta^2}{2}\left(H_{n-1}^{(1)}J_{n-1}+H_{n+1}^{(1)}J_{n+1}\right)\right]\right\} \tag{46}
$$

where the argument of the cylinder functions is $\sqrt{(n\beta)^2-(j\pi R/a)^2}$. Now if the argument is imaginary, $n\beta < j\pi R/a$, the product $H_m^{(1)}J_m$ is also imaginary, and such terms give no contributions to the radiated power. Thus, if $n < \pi R/\beta a$, the argument is imaginary for all j, and no radiation is emitted. If $3\pi R/\beta a > n > \pi R/\beta a$, only the single mode $j = 1$ is excited, and more generally, if $(2k+1)\pi R/\beta a > n > \pi R/\beta a$, radiation will be emitted into the first k modes. As an example, if $R = 5$ m, $a = 5$ cm, $\beta = 1$, no radiation is produced in the first 314 harmonics, and until the 953 harmonic, only a single mode of the parallel plate system is excited. Extracting the real part in Eq. (46), we get

$$
\begin{aligned}
P_n &= n\frac{\omega e^2}{R}\frac{4\pi R}{a}\sum_{\substack{j=1,3,\cdots \\ j<na\beta/\pi R}}\left[-J_n^2+\frac{\beta^2}{2}\left(J_{n-1}^2+J_{n+1}^2\right)\right] \\
&= n\frac{\omega e^2}{R}\frac{4\pi R}{a}\sum_{\substack{j=1,3,\cdots \\ j<na\beta/\pi R}}\left[\beta^2 J_n'^2+\frac{(j\pi R/a)^2}{(n\beta)^2-(j\pi R/a)^2}J_n^2\right]
\end{aligned} \tag{47}
$$

where the argument of the cylinder functions is the same as above. A simple check of this result is obtained by supposing the separation of the plates to become infinite, which enables the summation to be replaced by an integration. Writing $j\pi R/a = n\beta\cos\theta$, and noting that the interval between successive values of j is 2, we regain the formula for radiation by an electron in free space, expressed in the form (32).

The expression (47) for the power radiated by an electron into the n-th harmonic in the presence of a parallel plate metallic system can be accurately approximated by a simpler formula under the condition $\pi R/a \gg 1$, for the harmonics involved in the Bessel functions of large order are applicable. The formulas (27) and (28) are conveniently written, for this purpose, as

$$
J_n\left(\sqrt{n^2-\gamma^2}\right) = \frac{1}{\pi\sqrt{3}}\frac{\gamma}{n}K_{1/3}\left(\frac{\gamma^3}{3n^2}\right) \tag{48a}
$$

$$
J_n'\left(\sqrt{n^2-\gamma^2}\right) = \frac{1}{\pi\sqrt{3}}\left(\frac{\gamma}{n}\right)^2 K_{2/3}\left(\frac{\gamma^3}{3n^2}\right). \tag{48b}
$$

In the relativistic energy region, β may be replaced by unity, provided $n < n_0$, and the formulas (48) are immediately applicable to (47), with $\gamma = j\pi R/a$. In consequence of the properties of the functions $K_{1/3}(x)$ and $K_{2/3}(x)$, the Bessel functions occurring in (47) are negligible unless $n \gtrsim j\pi R/a\sqrt{j\pi R/a}$. Hence for a given mode of the parallel plate system, radiation is not excited appreciably unless the harmonic number n exceeds the critical value $j\pi R/a$ by a fairly large factor $\sqrt{j\pi R/a}$. On simplifying the second term of (47) in accordance with this observation, and replacing the Bessel functions by their approximate representations, we obtain

$$P_n = \frac{\omega e^2}{R}\frac{4R}{3\pi a}\sum_{\substack{j\,=\,1,3,\cdots\\ \gamma_j\,<\,n}}\frac{\gamma_j^4}{n^3}\left[K_{1/3}^2\left(\frac{\gamma_j^3}{3n^2}\right)+K_{2/3}^2\left(\frac{\gamma_j^3}{3n^2}\right)\right]$$

where $\gamma_j = j\pi R/a$.

To evaluate the total coherent power radiated by N electrons, as given by Eq. (39), we shall assume that the pulse length $R\alpha$ is of the same order of magnitude as the spacing of the plates. Hence, for those values of n at which radiation into the j-th mode becomes appreciable, $n\alpha/2 \sim (jR\alpha/a)\sqrt{j\pi R/a}$, which is a fairly large number. Consequently, we may consider $\sin^2 n\alpha/2$ as a rapidly oscillating function of n and replace $\sin^2 n\alpha/2$ by its average value, whence

$$P_{\text{coh.}}^{(N)} = N^2\frac{2}{\alpha^2}\sum_n\frac{P_n}{n^2} = N^2\frac{\omega e^2}{R}\frac{1}{\alpha^2}\frac{8R}{3\pi a}\sum_{j=1,3,\cdots}\sum_{n>\gamma_j}\frac{\gamma_j^4}{n^5}\left[K_{1/3}^2\left(\frac{\gamma_j^3}{3n^2}\right)+K_{2/3}^2\left(\frac{\gamma_j^3}{3n^2}\right)\right].$$

Since only large values of n are involved in the latter summation, we may replace the sum by an appropriate integral. The quantity $x = \gamma_j^3/3n^2$, as a function of n, varies from 0 to $\gamma_j/3$. However, the latter limit, being large in comparison with unity is effectively infinite, and therefore

$$\begin{aligned}P_{\text{coh.}}^{(N)} &= N^2\frac{\omega e^2}{R}\frac{1}{\alpha^2}\frac{12R}{\pi a}\left(\sum_{j=1,3,\cdots}\frac{1}{\gamma_j^2}\right)\int_0^\infty dx\,x\left(K_{1/3}^2(x)+K_{2/3}^2(x)\right)\\ &= N^2\frac{\omega e^2}{R}\frac{1}{\alpha^2}\frac{3a}{2\pi R}\int_0^\infty dx\,x\left(K_{1/3}^2(x)+K_{2/3}^2(x)\right).\end{aligned}$$

On employing the formula

$$\int_0^\infty dx\,xK_\nu^2(x) = \frac{\pi\nu}{2\sin\pi\nu}\,, \qquad \nu < 1$$

we finally obtain

$$P_{\text{coh.}}^{(N)} = N^2\frac{\omega e^2}{R}\frac{\sqrt{3}}{2}\frac{a}{R\alpha^2}$$

which can be described by the radiation resistance

$$\mathcal{R} = 120\pi^2\frac{\sqrt{3}}{\alpha}\frac{a}{2R\alpha} \quad \Omega$$

as compared with

$$\mathcal{R} = 120\pi^2\left(\frac{\sqrt{3}}{\alpha}\right)^{4/3} \quad \Omega$$

in the absence of the metallic plates. Hence, the insertion of the metallic plates has reduced the radiation by the factor $(\alpha/\sqrt{3})^{1/3}(a/2R\alpha)$. For numerical illustration, we choose the same numbers as before, $\alpha = 4\times10^{-2}$ and $R = 2.5$ m, corresponding to a pulse length of 10 cm. If $a = 5$ cm, the factor under discussion has the value 0.071 and the radiation resistance has been reduced to $\mathcal{R} = 1.3\times10^4$ Ω. For an average circulating current of 3 mA, the power appearing in coherent radiation is 0.12 W, and the coherent energy loss of an electron, per revolution, is 39 eV. Under these conditions, the incoherent radiation loss exceeds the coherent effect when the electron energy is greater than 1.8×10^8 eV. It hardly need be remarked that, at such energies, the incoherent radiation is unaffected by the presence of the metallic plates.

References

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[4] I am indebted to Prof. G. K. Uhlenbeck for this remark.

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