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A POSSIBLE SYMPLECTIC COHERENT BEAM-BEAM MODEL

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We consider a simple model to study the effects of the beam-beam force on the coherent dynamics of colliding beams. The key ingredient is a linearized beam-beam kick. We study only the quadrupole modes, with the dynamical variables being the 2nd-order moments of the canonical variables q, p. Our model is self-consistent in the sense that no higher-order moments are generated by the linearized beam-beam kicks, and that the only source of violation of symplecticity is the radiation. We discuss the cases of round and flat beams. Depending on the values of the tune and beam intensity, we observe steady states in which otherwise identical bunches have sizes that are equal, or unequal, or periodic, or behave chaotically from turn to turn. Possible implications of luminosity saturation with increasing beam intensity are discussed.

Introduction

The study of the coherent modes of oscillation of colliding beams has a long history, with many contributions to this important and difficult problem. Space limitations prevent us from giving here a full set of references. Recently Hirata [1] has studied the problem in a simplified model that includes coupled-beam features, but is inconsistent with Vlasov's equation. However, it explains qualitatively the "flip-flop" effect and the saturation of the luminosity and beam-beam parameter at high intensity. We summarize here the results of a simpler model [2], defined along similar lines, that has the virtue of being fully self-consistent (*i.e.*, symplectic in the absence of radiation, with Gaussian beams remaining Gaussian) since it involves the essential ingredient of a *linearized* beam-beam force. The consistency with Vlasov's equation is achieved at the price of ignoring Maxwell's equations altogether, since the force is assumed to be linear at all distances while the bunch size is finite. This is clearly not a good approximation for any reasonable distribution. However, since we study the quadrupole modes only, the linear part of the force has the most important effect, and in this sense it is reasonable to make such an approximation.

Model and Results

We consider a collider ring with tune ν, a single interaction point, and one bunch per beam. We call them as the e^+ and e^- bunches, although our discussion allows for like-charged beams ($e^\pm e^\pm$) just as well. We consider only the vertical dynamics described by y, y', and define the normalized coordinates q, p for a particle in each beam as $q_\pm \equiv y_\pm/\sqrt{\beta_y}$ and $p_\pm \equiv (\beta_y y'_\pm + \alpha_y y_\pm)/\sqrt{\beta_y}$. We represent the beam-beam interaction by the linearized kick

$$q'_\pm = q_\pm, \qquad p'_\pm = p_\pm - k_\mp q_\pm \tag{1}$$

This is the only source of coupling and of nonlinearity since the kick strength k depends inversely on the size of the other bunch, which is a *dynamical variable*. We consider the two extreme cases of "flat beam" and "round beam" shapes; in terms of the nominal beam-beam parameter ξ_0, we have

$$\begin{aligned} k_\pm &= 4\pi\xi_0\sqrt{\frac{\epsilon_{y0}}{\langle q_\pm^2\rangle}}, \qquad \xi_0 = \frac{r_0 N}{2\pi\gamma\sqrt{\epsilon_{x0}\epsilon_{y0}}}\sqrt{\frac{\beta_y}{\beta_x}} \quad \text{(flat beam)} \\ k_\pm &= 4\pi\xi_0\frac{\epsilon_0}{\langle q_\pm^2\rangle}, \qquad \xi_0 = \frac{r_0 N}{4\pi\gamma\epsilon_0} \quad \text{(round beam)} \end{aligned} \tag{2}$$

where ϵ_{x0} and ϵ_{y0} are nominal emittances (we assume $\beta_x = \beta_y \equiv \beta$ and $\epsilon_{x0} = \epsilon_{y0} \equiv \epsilon_0$ for the round-beam case).

Following Hirata [1], we represent the effect of radiation by the stochastic localized kick

$$q'_\pm = q_\pm, \qquad p'_\pm = \lambda p_\pm + \hat{r}_\pm\sqrt{\epsilon_{y0}(1-\lambda^2)} \tag{3}$$

where the $\hat{r}_\pm$ are independent random numbers with $\langle \hat{r}_\pm\rangle = 0$ and $\langle \hat{r}_\pm^2\rangle = 1$ and λ is related to the "damping decrement" δ by $\lambda = \exp(-2\delta)$.

The third ingredient is a linear transport through a phase advance $2\pi\nu$, given by the usual 2×2 matrix.

The bunches undergo collision, transport, radiation, collision, etc. The one-turn map for a given particle has a stochastic inhomogeneous part arising from the last term in (3). A deterministic map is obtained from it by taking the bunch-averages of the bilinear combinations of q and p and averaging these over all radiation events. With a surface of section just before the beam-beam kick we find

$$\begin{bmatrix} \langle q_+^2\rangle \\ \langle p_+q_+\rangle \\ \langle p_+^2\rangle \end{bmatrix}_{n+1} = \widetilde{M}(k_{-,n}) \begin{bmatrix} \langle q_+^2\rangle \\ \langle p_+q_+\rangle \\ \langle p_+^2\rangle \end{bmatrix}_{n} + \epsilon_{y0}(1-\lambda^2)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

where $\widetilde{M}(k_{-,n})$ depends on $\langle q_-^2\rangle_n$, ν, λ and ξ_0 (there is a simultaneous companion map with $+ \leftrightarrow -$).

Period-One Fixed Points

Setting $\langle\cdots\rangle_{n+1} = \langle\cdots\rangle_n$ for all six moments yields a set of two equations for $\langle q_+^2\rangle$ and $\langle q_-^2\rangle$. By defining $k_+ \equiv (\lambda+1)x$, $k_- \equiv (\lambda+1)y$, $\rho \equiv 4\pi\xi_0/(\lambda+1)$ and $\chi \equiv \cot(2\pi\nu)$, we obtain

$$(x/\rho)^s = 1 + 2\chi y - y^2, \qquad (y/\rho)^s = 1 + 2\chi x - x^2 \tag{5}$$

where $s = 2$ for flat beams, and $s = 1$ for round beams. Eq. (5) admits $x = y$ ("normal") and $x \neq y$ ("flip-flop") solutions, which can be found analytically in a straightforward way. They correspond to equal-size and unequal-size beams. Note that they depend on ν and ρ, but do not depend separately on λ. In order to be physical, the solutions must be real and have the same sign. The normal solutions are always real: the + solutions are physical for e^+e^-, the − solutions for $e^\pm e^\pm$. The flip-flop solutions are physical only in certain regions of the $\nu - \rho$ plane, which are shown shaded in Figs. 1 and 2. Note that round beams do not admit $e^\pm e^\pm$ flip-flop solutions. In addition, the solutions must

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be stable. This is determined from the 6×6 stability matrix, obtained by expanding the map infinitesimally close to the fixed point. Results for the size and stability of the e^+e^- case are shown in Figs. 3 and 4 for flat and round beams, respectively.

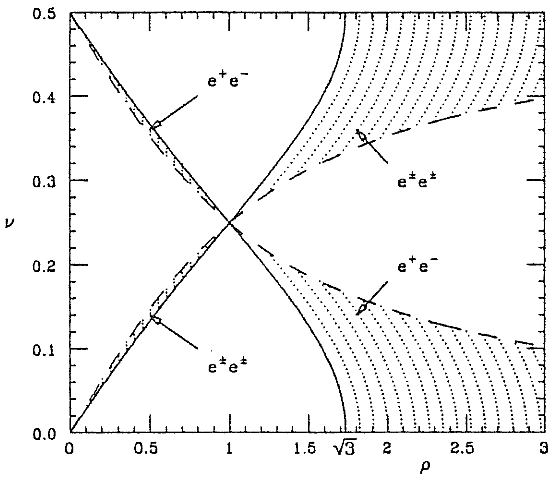

Fig. 1. Regions where the flat-beam, period-1, unequal-size solutions are real (though not necessarily stable).

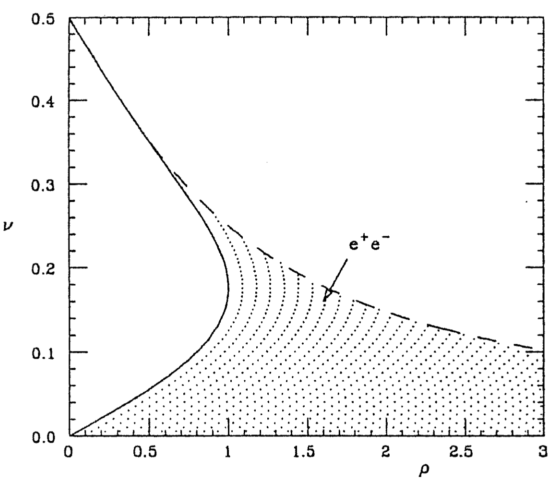

Fig. 2. Region where the round-beam, period-1, unequal-size solutions are real (though not necessarily stable). The $e^\pm e^\pm$ solutions are always complex.

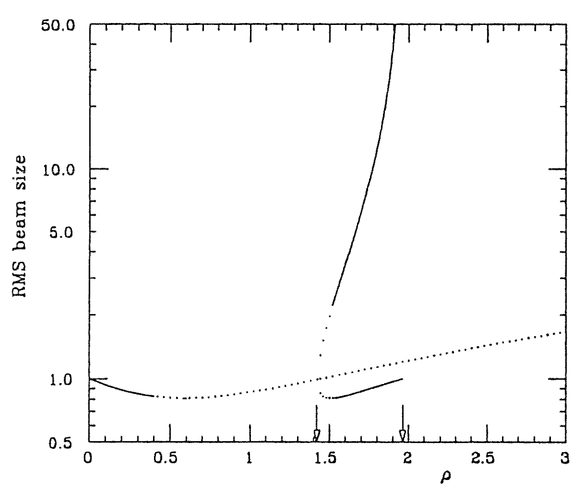

Fig. 3. RMS beam sizes and stability for the period-1 fixed point solutions (solid=stable, dots=unstable). Flat-beam case, ν=0.15, λ=0.8694.

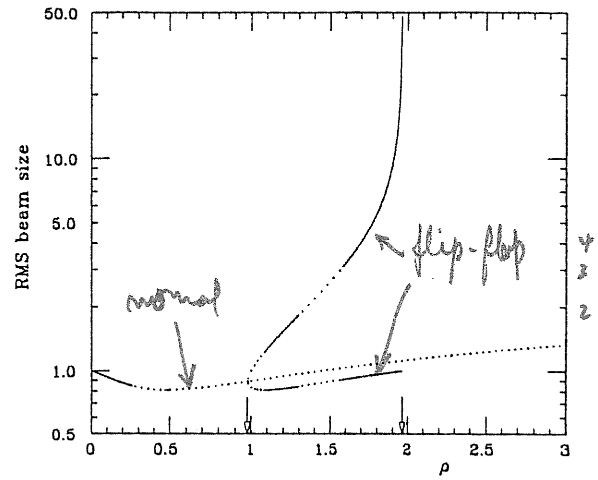

Fig. 4. RMS beam sizes and stability for the period-1 fixed point solutions (solid=stable, dots=unstable). Round-beam case, ν=0.15, λ=0.8694.

Iteration of the Map

By starting with a given set of values for the moments we iterate the map (4) until it converges or diverges. All results presented here are for e^+e^-, for $\nu = 0.15$ and $\delta = 0.07$ ($\lambda = 0.8694$). This is an unrealistically large value of δ; however, because our model is symplectic in the absence of radiation, all our results have a smooth $\lambda \to 1$ limit, and are *quantitatively similar* for any λ sufficiently close to 1 (a large δ has the practical advantage of fast convergence of the map iteration). Results are shown in Figs. 5 and 6; dots represent chaotic behavior, in which the two beams are preferentially of different size; + represents period-1 fixed points, in which the beams are of equal or unequal size, depending on the value of ρ (they correspond to the beam sizes shown in Figs. 3 and 4); $\times$, $\diamond$ and $\square$ represent period-2, -3, and -4 fixed points with beams of equal size. Other types of solutions may well exist, but are hard to find. If more than one solution is possible, the one to which the map converges depends on the initial conditions. For $\rho \simeq 0.5$ for the flat-beam case, and $\rho \simeq 0.3$ for the round-beam case, the chaotic solutions are the most stable. For other values of ρ, generally speaking, the period-1 fixed point is the most stable unless it coexists with higher-order fixed points. In this case the period-2 fixed point is the most stable for the flat-beam case, while the period-3 fixed point is the most stable for the round-beam case. By "most stable" we mean that this solution is the most likely to be reached when varying the initial conditions.

The effects of the map can be evaluated by looking at "observables" such as the luminosity or the effective beam-beam parameter, which depend on the actual emittance. Thus a quantity that measures the physical effects of our model is the "enhancement factor" E defined by $E \equiv \mathcal{L}/\mathcal{L}_0 = \xi/\xi_0$,

$$E = \sqrt{\frac{\epsilon_{y0}}{\epsilon_y}} = \sqrt{\frac{2\epsilon_{y0}}{\langle q_+^2 \rangle + \langle q_-^2 \rangle}} \qquad \text{(flat beam)}$$
$$E = \frac{\epsilon_0}{\epsilon} = \frac{2\epsilon_0}{\langle q_+^2 \rangle + \langle q_-^2 \rangle} \qquad \text{(round beam)} \tag{6}$$

Figs. 7 and 8 show E *vs.* ρ for flat and round beams respectively (we compute its average over the period of the most stable fixed point). Note the saturation effect due to chaotic behavior and higher-order fixed points.

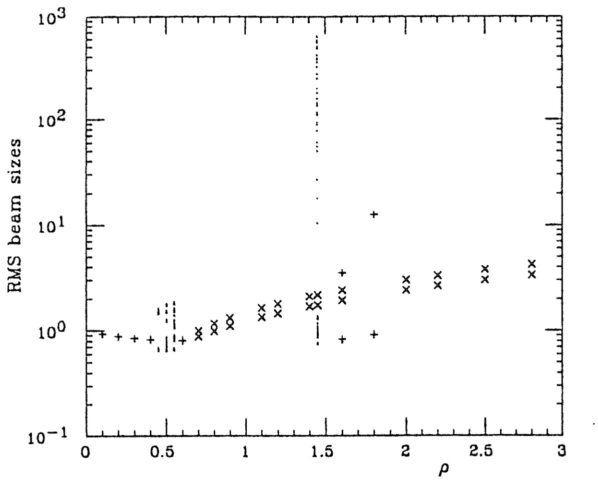

Fig. 5. Flat beam sizes from map iteration; dots=chaotic; +=period−1 (equal or unequal sizes); ×=period−2 (equal sizes). ν=0.15, λ=0.8694.

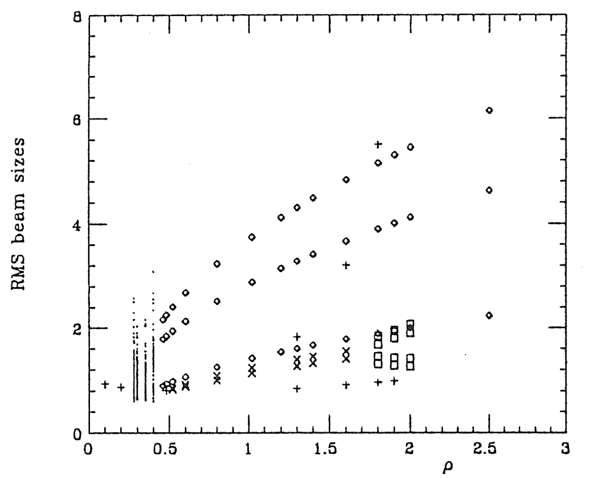

Fig. 6. Round beam sizes from map iteration; dots=chaotic; +=period−1 (equal or unequal sizes); ×=period−2, ◇=period−3, □=period−4 (equal sizes). ν=0.15, λ=0.8694.

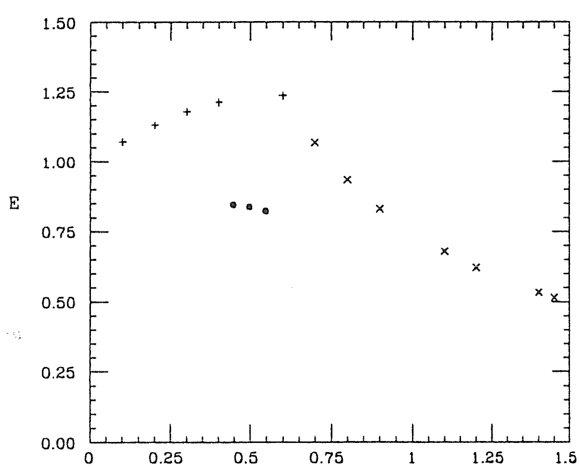

Fig. 7. Flat-beam enhancement factor from period−1 (+), period−2 (×), or chaotic (⊛) fixed points. ν=0.15, λ=0.8694.

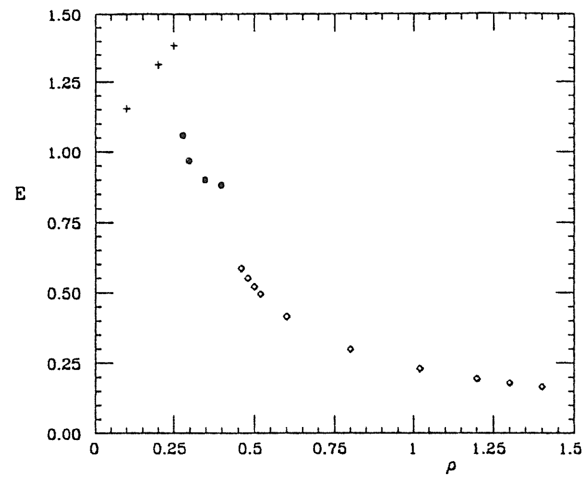

Fig. 8. Round−beam enhancement factor from period−1 (+), period−3 (◇), or chaotic (⊛) fixed points. ν=0.15, λ=0.8694.

Conclusions

(1) For low beam intensity (small ρ), only normal solutions exist and are stable.

(2) As the intensity is increased, other solutions appear which cause the saturation of the luminosity and beam-beam parameter at the realistic values $\xi_0 \simeq \rho/2\pi \simeq 0.065$ for flat beams and $\simeq 0.043$ for round beams. The saturation mechanism is due to the appearence of a chaotic region followed by a higher-order fixed point rather than to a bifurcation. This seems to be a key difference with Hirata's result [1].

(3) Flip-flop solutions exist and are real for all values of ρ. However, they are not always stable and are unnatural for small ρ. By this we mean that they require a delicate relationship between ν and ρ, as can be seen from Figs. 1 and 2. Therefore the flip-flop effect may have a natural explanation in our model for unrealistically large beam intensity.

Acknowledgements

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References

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